THE JUNE MEETING IN SEATTLE

The three hundred thirty-third meeting of the American Mathematical Society was held jointly with the meeting of the Pacific Division of the American Association for the Advancement of Science, at the University of Washington, Thursday, June 18. At the time of this meeting, the University of Washington was holding a celebration to commemorate its seventy-fifth anniversary and, as a part of this celebration, the Department of Mathematics held a reunion of its alumni under the joint auspices of the American Mathematical Society and the University of Washington. As a feature of this joint meeting, three alumni of the University of Washington gave invited addresses as follows: Professor H. P. Robertson, *Geometry and physical space-time*; Professor E. T. Bell, *Highlights of mathematical biography*; Professor Harold Hotelling, *Correlated vectors*. The morning session, presided over at different times by Professor A. F. Carpenter and Professor E. T. Bell, was devoted to the presentation of the program of the American Mathematical Society, and was concluded by the address of Professor Robertson. The afternoon session was presided over by Professor Milne, of Oregon State College. The features of this session were the addresses by Professors Bell and Hotelling.

Luncheon for members and their guests was held at the Seattle Yacht Club, during which Professor A. R. Jerbert read a brief history of the Department of Mathematics of the University of Washington.

About fifty persons were in attendance at the meeting, including the following twenty-three members of the Society:

H. C. Ayres, J. P. Ballantine, E. T. Bell, A. F. Carpenter, C. M. Cramlet, W. H. Gage, F. L. Griffin, I. M. Hostetter, Harold Hotelling, Susanne E. Hotelling, R. D. James, A. R. Jerbert, J. C. Knipp, D. H. Lehmer, L. H. McFarlan, W. E. Milne, L. I. Neikirk, E. D. Pepper, H. P. Robertson, L. L. Smail, J. C. Trainor, R. M. Winger, Frederick Wood.

The titles of the papers read at the meeting follow. Those whose abstract numbers are followed by the letter *t* were read by title. Dr. Mary E. Haller was introduced by Professor Winger, Mr. I. E. Highberg by Professor Michal, and Mr. P. O. Bell by Professor Pauline Sperry.

1. *Problems on the borderline between mathematics and psychology*, by Professor J. C. Trainor. (Abstract No. 42-5-242.)

2. *Self-projective rational octavics*, by Dr. Mary E. Haller. (Abstract No. 42-5-249.)

3. *On functions possessing polynomial addition theorems*, by Dr. D. H. Lehmer. (Abstract No. 42-5-251.)

4. *A vector treatment of the first necessary condition in the problem of Lagrange with finite equations*, by Dr. I. M. Hostetter. (Abstract No. 42-5-253.)

5. *Linear differential equations with constant coefficients*, by Professor C. M. Cramlet. (Abstract No. 42-7-256.)

6. *Geometry and physical space-time*, by Professor H. P. Robertson. (Abstract No. 42-7-258.)

7. *Special forms of the Euler differential equations when x is absent from the integrand*, by Professor L. H. McFarlan. (Abstract No. 42-7-259.)

8. *Arc cotangent triads*, by Professor J. P. Ballantine. (Abstract No. 42-7-260.)

9. *Some symbolic identities*, by Professor L. I. Neikirk. (Abstract No. 42-7-264.)

10. *An extension of two formulas of Hurwitz*, by Dr. R. D. James. (Abstract No. 42-7-265.)

11. *A note on pseudo-polynomials in abstract spaces*, by Mr. I. E. Highberg. (Abstract No. 42-7-268.)

12. *A note on association by quartic matric polynomials*, by Mr. Ivan Niven and Professor F. S. Nowlan. (Abstract No. 42-7-269.)

13. *Covariant configurations associated with a general curved surface at one of its points*, by Mr. P. O. Bell. (Abstract No. 42-7-270.)

14. *Self-corresponding line complexes*, by Professor A. F. Carpenter. (Abstract No. 42-5-246-*t*.)

15. *On measures and weights by Epiphanius*, by Professor A. A. Shaw. (Abstract No. 42-5-247-*t*.)

16. *A cubic analog of the Cauchy-Fermat theorem*. II, by Mr. Alvin Sugar. (Abstract No. 42-5-243-*t*.)

17. *Proof of the non-isomorphism of two collineation groups of order* 5184, by Professor F. A. Lewis. (Abstract No. 42-5-244-*t*.)

18. *Linear divisibility sequences*, by Professor Morgan Ward. (Abstract No. 42-5-245-*t*.)

19. *Note on Taylor's theorem*, by Professor R. E. Moritz. (Abstract No. 42-5-248-*t*.)

20. *An algorithm associated with the cotangent function*, by Dr. D. H. Lehmer. (Abstract No. 42-5-250-*t*.)

21. *An extension of Gibbs' vector analysis to n-space*, by Dr. I. M. Hostetter. (Abstract No. 42-5-252-*t*.)

22. *Continuity in topological groups*, by Dr. Deane Montgomery. (Abstract No. 42-5-255-*t*.)

23. *Pointwise periodic homeomorphisms*, by Dr. Deane Montgomery. (Abstract No. 42-5-254-*t*.)

24. *On the reduction of a representation to classical canonical form*, by Professor C. M. Cramlet. (Abstract No. 42-7-257-*t*.)

25. *Concerning more general topologically flat spaces*, by Dr. F. B. Jones. (Abstract No. 42-7-261-*t*.)

26. *Certain equivalences and subsets of a plane*, by Dr. F. B. Jones. (Abstract No. 42-7-262-*t*.)

27. *On the forces occurring in a dynamical system where the coordinate axes have variable relative obliquity* (preliminary report), by Mr. W. H. Ingram. (Abstract No. 42-7-263-*t*.)

28. *On a theorem of Lie*, by Dr. Max Zorn. (Abstract No. 42-7-266-*t*.)

29. *Nilpotency of finite groups*, by Dr. Max Zorn. (Abstract No. 42-7-267-*t*.)

30. *Existence and embedding theorems for hyperbolic systems of partial differential equations*, by Dr. H. C. Ayres. (Abstract No. 42-7-272-*t*.)

In the absence of Associate Secretary T. M. Putnam, Dr. R. D. James acted as secretary.

T. M. PUTNAM,
Associate Secretary

WALSH ON APPROXIMATIONS

Interpolation and Approximation by Rational Functions in the Complex Domain. By J. L. Walsh. American Mathematical Society Colloquium Publications, vol. 20. New York, 1935. ix+381 pp.

In 1885, C. Runge, the outstanding exponent of applied mathematics in Germany, published a theorem concerning analytic functions which became the starting point of a long and fruitful evolution. It states, as does a classical theorem of Weierstrass about continuous functions of a real variable, the existence of polynomials approximating with a preassigned accuracy a given analytic function, regular in the closed interior of a curve. Runge's proof, although very elementary and ingenious, does not give a definite principle (see below) by which the approximating polynomials can be calculated, or, to use an equivalent formulation, by which the development of the given function in a polynomial series can be obtained. Later on various other proofs have been given, characterizing the polynomials in question by means of important additional properties. Among others, developments of the form

$$c_0p_0(z)+c_1p_1(z)+c_2p_2(z)+\cdots+c_np_n(z)+\cdots$$

were investigated, in which the polynomials $\{p_n(z)\}$ depend only on the curve considered and the coefficients $\{c_n\}$ on the individual function developed. An account of the older results in this direction was given by P. Montel in his book, *Leçons sur les Séries de Polynomes à une Variable Complexe* (Borel's Collection, 1910). Since that time, however, a further extremely intensive and in some respects very fascinating cultivation of this field can be observed. A short survey of the modern situation was given by Walsh in a small volume in the French series *Mémorial des Sciences Mathématiques* (vol. 73, 1935). The present book of Walsh, which appeared as the latest volume of the Colloquium Lectures, takes up the recent development on a wide basis. Walsh himself is participating in the advancement of this field to a considerable extent, partly by various investigations of his own, partly by his suggestions to other writers. He gives in this book not only a careful and detailed picture of the present stage but furnishes also some new contributions to the subject.

Generally speaking this book deals primarily with infinite sequences or series of polynomials and rational functions converging towards a given analytic function and fulfilling some additional conditions. These conditions are characterized either by the coincidence of the polynomial or rational function with the given analytic function in proper points (problem of interpolation) or lying as "close" to it as possible (problem of the best approximation). Moreover the connection of those interpolatory and approximative properties is investigated. The theory of interpolation and the theory of approximations are, of course, tremendously broad fields. Among others they have a branch in the pure real region taking up the properties of sequences of polynomials associated with a continuous function of a real variable possessing a given

"degree" of continuity (S. Bernstein, de la Vallée Poussin, Jackson (see vol. 11 of these Colloquium Lectures)). The present book has no direct connections with these ideas. It considers sequences belonging exclusively to analytic functions. Furthermore, speaking again roughly, it investigates infinite sequences of polynomials (rational functions) convergent with the rapidity of the geometric series or having bounds of the form $O(R^n)$, R a fixed positive number, $n \to \infty$.

In order to give a more exact idea of the material treated in the book, we refer briefly to some more or less central results of this field.

The first proof of Runge's theorem associating a definite type of polynomial development with an analytic function is that given by Hilbert (1897). He uses to this end the so-called Jacobi series, proceeding in terms of the powers of a fixed polynomial $p(z)$, say of kth degree, and having coefficients which are polynomials of a degree $\leqq k-1$. Here $p(z)$ depends only on the given curve, the coefficients, however, on the individual function developed. In the case of a circle $p(z)$ is linear, and we obtain the customary power series development. Apart from the elegance of this result, Hilbert's proof was destined to influence the later evolution primarily because of another reason. It displays for the first time the connection of the general problem with potential theory and thus virtually also with conformal representation. He obtains indeed the polynomial $p(z)$, starting from the classical equilibrium problem of the logarithmic potential, dealt with by Gauss, and approximating the so-called "natural mass distribution" (possessing a constant potential interior to the curve) by means of discrete masses.

Faber was the first (1903) to use particular polynomials connected with the conformal representation of the region exterior to the given curve onto the exterior of a circle conserving the point at infinity. These polynomials can be calculated easily from the map function. Faber proves then the possibility of developing a given analytic function in a series of the form above indicated, $p_n(z)$ being the polynomials in question.

Fejér considers (1918) polynomials coinciding with a given analytic function, regular interior to and on a curve, in "uniformly distributed" points of this curve. The last term is meant in the sense of the conformal representation described before; the points in question correspond to points uniformly distributed on the circle in the ordinary sense. By previous important results of Runge (1903) it was well known that subdivision of the given curve in *equal arcs* does not lead in general to a convergent interpolation procedure. Fejér's subarcs are, indeed, equal in the sense that they contribute *equal masses* to the equilibrium or natural distribution mentioned above. Incidentally, Kalmar showed that this condition, at least in the asymptotical sense, is also necessary to the convergence of the interpolation procedure.

Faber (1920) investigates polynomials of a given degree, differing in the sense of Tchebycheff as little as possible from the given function (Tchebycheff polynomials). These polynomials have a simple asymptotic relation to the map function. Their explicit determination is however not possible, except in special cases.

Szegö introduces (1921) the corresponding polynomials in the sense of Bessel's approximation (orthogonal polynomials). They possess asymptotic

properties similar to those of the Tchebycheff polynomials and can be calculated explicitly by means of Gram's orthogonalization procedure.

Fekete (1926) considers the product of all $C_{n,2}$ distances of n points, running independently of one another on the given curve. The system of n points giving a maximal value to this product, $n=2, 3, 4, \cdots$, can be used in the same way as in Fejér's case in place of the "uniformly distributed" points as interpolation points of a convergent procedure.

This short enumeration of special sequences and developments associated with a given analytic function illustrates the colorful variety of problems arising here. The reader finds in the book of Walsh not only an exhaustive treatment of these questions but also important refinements, generalizations, and analogs which are sometimes rather intricate and require great care.

Concerning the details the following indications may be sufficient. Chapters 1 and 2 start with various preparations from the theory of point sets and analytic functions. We find here the formulation of Runge's theorem and its extensions, Lindelöf's map function theorems, remarks concerning Cauchy's integral, approximations in general and related topics. Chapters 3 and 4 take up lemniscates, Jacobi series, and Green's functions, altogether the background of Hilbert's proof in the refined shape of modern analysis. In the subsequent Chapters 5 and 6 the best approximation of a given analytic function is treated, primarily in the sense of Tchebycheff and Bessel. The last problem is, of course, intimately connected with convergence in the mean. We find here a short treatment of the Fischer-Fr. Riesz theorem. Numerous generalizations of the Tchebycheff and orthogonal polynomials (in particular introducing a "norm" function on the curve) can be likewise defined. Chapter 7 is devoted to the problem of interpolation in "uniformly distributed" points. The questions of interpolation and approximation are revised in Chapters 8 and 9 replacing polynomials by rational functions with preassigned poles or with preassigned restrictions concerning the number and position of the poles. Lemniscates and Green's functions are here used again, in a more general form. The next chapter studies analytic functions regular in the unit circle and fulfilling there some interpolatory conditions. The problem is, roughly speaking, to find the function of this type having a maximum modulus (or square integral) as small as possible. A long list of investigations dealing with this and similar questions could be enumerated. The recent results of Nevanlinna and Denjoy are here presented. Chapter 11 takes up interpolation by polynomials under auxiliary conditions. The last chapter gives a survey of existence and uniqueness theorems in the case of Tchebycheff and other approximations.

The bibliography is fairly complete. The reviewer missed among others a reference to one of his own papers: *Bemerkungen*, $\cdots$ (Mathematische Zeitschrift, vol. 21 (1924), pp. 203–208). This contains not only a general proof of the coincidence of the transfinite diameter with the capacity but also the extension of the conception of uniformly distributed points to the case of a finite number of mutually exterior analytic curves.

The exposition of the book is clear, the form elegant, the style lucid and plain, the presentation careful and detailed, now and then almost too detailed. The reader may have the feeling that the author was sometimes hesitating as

to whether the book should be destined for the beginner or whether its purpose should rather be a survey of a field for experts and fellow-contributors. A little contrast can be doubtless observed between the broad treatment of some comparatively elementary situations and the preference of special and individual topics to some others of importance. But at all events the book will be useful from both points of view and it is a significant enrichment of the series of the Colloquium Lectures.

GABRIEL SZEGÖ

MORSE ON CALCULUS OF VARIATIONS

The Calculus of Variations in the Large. By Marston Morse. Colloquium Publications of the American Mathematical Society, vol. 18. New York, 1934. ix+368 pp.

The background for the theory elaborated in this volume lies in two rather distinct fields of mathematics. We have on the one hand the theory of critical points of functions of n real variables, largely created and developed by the author and his students; on the other hand, the classical calculus of variations and its modern treatment as a part of the functional calculus, to which Hadamard and Tonelli have made the fundamental contributions. The origin of the theory of critical points is perhaps to be found in the minimax principle introduced by Birkhoff in his paper on *Dynamical systems with two degrees of freedom* (see Transactions of this Society, vol. 18 (1917), p. 240). While at this early stage the connection of the theory with the calculus of variations was made clear, later developments have made the relations between the two fields much more intimate.

The calculus of variations in the large derives its interest not only from its use of functional calculus methods. More significant is the fact that it undertakes a study of the configurations to which a calculus of variations problem gives rise, namely, the extremals, with respect to important properties apart from the question whether or not they furnish extreme values for a definite integral. In the classical calculus of variations, only such extremal arcs AB were considered which contained no point A' conjugate to A. The theory with which this book is concerned, and also the earlier investigations of Birkhoff, reveal the importance of extremal arcs AB upon which there may be one or more points conjugate to A; not only arcs in the small, restricted by the exclusion of conjugate points, but also arcs in the large have to be considered for a full understanding of this theory.

A critical point x_i^0 of $f(x_1, \cdots, x_m)$ is a point at which all the first order partial derivatives of f vanish. Each critical point has a type number equal to the number of negative terms in the quadratic form which represents the second order terms of $f(x_i)-f(x_i^0)$, when reduced to a sum of squares. The author's earlier papers establish a beautifully simple set of relations between the numbers of critical points of various types of $f(x_1, \cdots, x_m)$ and the connectivity numbers of the domain R over which the function ranges. Thus the topological character of the domain of the independent variable is linked up

with the critical values of the function. In the present volume these relations are extended to a functional domain. Instead of a function $f(x_1, \cdots, x_m)$, we consider now a functional $J=\int f(x, y_i, y_i')dx$, depending upon a curve $y_i(x)$, $(i=1, \cdots, m, a \leqq x \leqq b)$, that is, upon a point in a function space. In place of the n-dimensional region R of euclidean space we have a domain in function space. The critical points are now the extremals of the functional, that is, the solutions of class C'' of the Euler equations connected with this functional. It becomes necessary to set up for domains in function space topological invariants in terms of which their connectivities can be defined. We shall now pass on to a more detailed consideration of the separate chapters.

Chapter 1 (pp. 1–17) first develops the classical theory of the integral $\int f(x, y_i, y_i')dx$, for fixed endpoints, not including the theory of the second variation. The conjugate point theory is developed in considerable detail in accordance with the important role it plays in later parts of the book. Necessary conditions for a weak minimum are derived, and sufficient conditions for a strong minimum. A neat proof of DuBois-Reymond's lemma appears on page 3.

Chapter 2 (pp. 18–36) takes up the problem with variable endpoints in the form used in several of the author's papers: to minimize

$$\theta(\alpha)+\int_{x^1(\alpha)}^{x^2(\alpha)} f(x, y_i, y_i')dx.$$

The discussion of the second variation introduces us to the author's methods. The "accessory boundary problem" associated with the second variation leads to the characteristic roots, to the index of an extremal, and to the formulation of sufficient conditions in terms of characteristic roots.

In Chapter 3 (pp. 37–79) we come to the study of extremal arcs which are not restricted to be free from conjugate points. The connection between conjugate points and characteristic roots is developed further and linked up with the important concept of the index form. This is accomplished by means of the method of broken extremals. A set of points $A_1, \cdots, A_p$ is taken on the extremal g, determined by $y_i=g_i(x)$; through each point A_i of this set a manifold is constructed, on which a point P_i is then taken. If the points P_i are sufficiently near the points A_i, two successive ones can be joined by extremal arcs of the functional

$$J^\lambda=\theta(\alpha)+\int_{x^1(\alpha)}^{x^2(\alpha)} [f(x, y_i, y_i')-(\lambda/2)\sum(y_i-g_i(x))^2]dx.$$

The value of this functional taken along the "broken extremal," that is obtained in this way, gives rise to a function of the coordinates of the points P_i. The second order terms in the expansion of this function yield a quadratic form with coefficients depending on λ, called the index form of g. Its properties play an important part in later work. Necessary conditions and sufficient conditions for a minimum are given in terms of the index form for the problems

with one and with two variable endpoints, and for periodic extremals. The index of a periodic extremal is evaluated in terms of conjugate points and of a numerical invariant, called the "order of concavity"; a necessary condition that a periodic extremal give a strong minimum is that the order of concavity be zero.

The work of the preceding chapters has shown the importance of self-adjoint boundary problems for the calculus of variations; Chapter 4 (pp. 80–106) is devoted to a systematic exposition of their theory. First comes an invariant formulation of self-adjoint boundary conditions for self-adjoint systems of differential equations of the second order in n variables. Then follows a discussion of boundary problems which involve a parameter, characteristic roots and solutions, the comparison of two problems which differ only in the boundary conditions, a general oscillation theorem, comparison of boundary problems involving distinct differential forms, and related topics. The chapter contains a large number of interesting and important theorems.

Chapter 5 (pp. 107–141) brings a study in tensor form of the parametric problem on a Riemannian manifold whose points and neighborhoods are 1–1 images of an m-dimensional simplicial circuit, a neighborhood of each point being "homeomorphic with a neighborhood of a point (x) in a euclidean m-space of coordinates" $x^1, \cdots, x^m$. For each neighborhood there is "at least one such representation" in which there is defined a positive definite form such as $g_{ij}(x)dx^i dx^j$, "defining a metric for the neighborhood". In a preliminary section there is a proof of the theorem on "normal coordinates" which justifies the representation of a "simple regular arc of class C^{1}" and a neighborhood of such an arc by a segment of a straight line in a euclidean space and a neighborhood of this segment. The theory, as developed in the earlier chapters for the non-parametric case, is here presented in an elegant and concise manner for the parametric problem $J=\int f(x, x_i)dt$. The theory of the index form and its connection with conjugate points and focal points is carried over to this problem by the use of the "normal coordinates".

The chapters whose contents have been briefly sketched thus far have introduced us to the new concepts which are more fully developed in the remaining chapters. Chapter 6 (pp. 142–192) deals with the theory of critical points of a function f on a Riemannian manifold R of the character introduced in the preceding chapter, so that in every neighborhood it can be represented as a function x of class C_2 of local coordinates $x_1, \cdots, x_n$. A point at which the first partial derivatives of x vanish is a critical point; the critical point is degenerate or non-degenerate according as the Hessian of x does or does not vanish there. If all the critical points are non-degenerate, they are isolated and hence finite in number; the function f is then called non-degenerate. In various papers the author has established the fundamental relations

$$\sum_{i=0}^{k}(-1)^{i+k}M_i \geqq \sum_{i=0}^{k}(-1)^{i+k}R_i, \qquad (k=0, 1, \cdots, n-1),$$

and

$$\sum_{i=0}^{n}(-1)^{i}M_i=\sum_{i=0}^{n}(-1)^{i}R_i,$$

which hold between the numbers M_i of critical points of type i of a non-degenerate function f and the connectivities R_i of R. The essentially new problem of the present chapter is that of the extension of these relations to the case in which degenerate critical points occur. It requires a study of critical sets σ, that is, sets of critical points at which f takes on a fixed value c. Corresponding to each such set, a set of type numbers m_k is introduced in terms of "new k-cycles" and of "newly-bounding k-cycles" and the critical set σ is said to be "equivalent" to m_k non-degenerate critical points of type k, for $k=0, \cdots, n$. The numbers m_k are completely determined by the definition of f in an arbitrarily small neighborhood of the set σ. The equivalence definition maintains the fundamental relations in the case of degenerate critical points. If f is analytic and is continued by a function Φ of the form $F(x, \mu)$ in the local coordinates on R and a parameter (μ), analytic and non-degenerate, reducing to f for $(\mu)=(0)$, then Φ will have at least m_k non-degenerate critical points of type k, near the critical set σ, provided (μ) is sufficiently near (0). It is with the proofs of these various statements that the chapter is chiefly concerned. They involve detailed topological considerations not amenable to a condensed form of statement. The chapter ends with applications of the theory to theorems on the number of normals from a point O to a manifold R in euclidean space and on the number of chords of R which are normal to R at both ends.

The methods developed in Chapter 6 are carried over in the next chapter (pp. 193–249) to the functionals of the calculus of variations discussed in Chapter 5. The independent variable now ranges over the domain Ω of curves γ, which are continuous images on R of the segment $0 \leqq t \leqq 1$, are of class D' on R, and have endpoints belonging to a terminal domain Z. The various topological concepts of the preceding chapter are defined in Ω. Between two points A^1 and A^2 of R, a set of points $P^1, \cdots, P^p$ is taken such that any two successive points of the set $A^1, P^1, \cdots, P^p, A^2$, denoted by π, can be joined by an "elementary extremal," that is, an extremal segment for which the integral J is less than a prescribed amount. The value $J(\pi)$ of the integral J taken along the broken extremal $g(\pi)$ constructed in this way is "a function ϕ of the parameters (α) locally representing its vertices A^1, A^2 and of the successive sets of coordinates (x) locally representing its vertices P^j". The critical points and critical sets of ϕ form the basis of the discussion and the connecting link between the earlier theory of critical points and the calculus of variations. A necessary and sufficient condition that a set (π), of which no two consecutive vertices coincide, be a critical point is that $g(\pi)$ be a "critical extremal". The topology of the critical extremals is investigated, resulting in theorems concerning the connectivities of Ω and of its subdomains determined by inequalities of the form $J(\pi)<b$. Deformations on Ω and a J-distance between two curves of class D' play an important part in these investigations. The details are too numerous to make a brief account feasible. They are concerned with isolated critical extremals and with critical sets of extremals. For a critical set, type numbers m_k are defined and it is shown that the sums N_k of the kth type numbers of all critical sets and the connectivities P_k of the domain Ω are connected by the inequalities $N_k \geqq P_k$. If $N_i (i=0, 1, \cdots, r)$, are finite, the inequalities of Chapter 6 are shown to hold for $k=0, 1, \cdots, r$. For a non-degenerate critical

extremal of index k, all the type numbers vanish except the kth, which is equal to 1. It follows that in the non-degenerate case "the number N_k of distinct extremals of index k" and the connectivities P_k of Ω satisfy the relations $N_k \geqq P_k$. After a further analysis of the non-degenerate problem, the problem with fixed endpoints and that with one variable endpoint are taken up. Application of the general theorems leads to conclusions concerning the number of conjugate points on extremals joining two points (Theorem 13.3, p. 239) and to certain corresponding results for focal points (Theorem 14.4, p. 244). The chapter closes with a discussion of the connectivities of the functional domain $\Omega(A^1, A^2)$ determined by the fixed-points A^1 and A^2 on an m-sphere. From the known character of the geodesics on a sphere it is easy to derive, by use of general theorems established in this chapter, the fact that the connectivities P_k of this domain are all 0 except those for which $k = p(m-1)$, $(p = 0, 1, \cdots)$; for these values of k, $P_k = 1$.

A new topological approach is required for the problem of closed extremals treated in Chapter 7 (pp. 250–306). The closed broken extremal with arbitrarily many vertices, taken in circular order, is the basal element; the set of all spaces representing such sets of vertices is the domain Ω whose connectivities are to be used. The central problem consists in the definition of chains, cycles, homologies, and connectivities in this space. This is done in first instance by means of a metric dependent on "elementary extremals". Then, by means of an abstract formulation of a "metric M_0 with elementary arcs" it is shown that the connectivities as defined for the space Ω are independent of the particular metric used. In addition to the usual postulates for a metric space S constituted by points $P, Q, R, \cdots$, two further postulates are used. After the uniform continuity of the distance PQ on S has been established and a continuous map of one such space on another has been defined, the term "simple arc γ" on S is introduced to designate the homeomorph of the segment $0 \leqq t \leqq 1$. The additional postulates are to the effect that for every space S there shall exist a positive constant ρ, so that for every pair of points P and R for which $0 < PR \leqq \rho$ there shall exist one simple arc $[PR]$ of such nature that for Q on $[PR]$, and between P and R, $PR = PQ + QR$, while for any point Q not on R, the inequality $PR < PQ + QR$ shall hold. The topological machinery that is obtained in this way is adequate to the needs of the problem. The connectivities in this space of sets of points in circular order are called circular connectivities. Theorems analogous to those obtained in the previous chapter are now proved, thus establishing for closed extremals fundamental relations between the circular connectivities of Ω and the index numbers of the extremals.

The final chapter (pp. 307–358) is concerned with a generalization of Poincaré's theorem concerning the existence of closed geodesics on a convex surface belonging to a family S_α of surfaces, containing an ellipsoid of unequal axes for $\alpha = 0$ and depending analytically on α. The author asks for "numerical characteristics of sets of closed geodesics on $R \cdots$ sufficient to guarantee the existence of a corresponding set" of closed geodesics on any other admissible Riemannian manifold homeomorphic with R. He proves that these sets of geodesics vary analytically with respect to any parameter with respect to which the manifolds vary analytically. The determination of these character-

istics for geodesics on the m-sphere furnishes criteria for the existence of closed extremals corresponding to a large class of functionals defined on manifolds homeomorphic to a sphere. The method used here is contrasted with Birkhoff's method of fixed points of transformations. The circular connectivities introduced in the preceding chapter are now determined for the m-sphere by means of an analysis of the closed geodesics on ellipsoids. The analysis is full of interesting details, it utilizes the topological concepts defined in earlier parts of the book, and leads to the following simple result. The kth circular connectivity of the m-sphere is the number of distinct integral solutions, i, j, r of the diophantine equation $k=m+i+j-4+2(r-1)(m-1)$ under the restrictions $m+1 \geqq i>j>0$, $r>0$. This result leads to existence theorems for closed extremals, of which the following is an example (Theorem 9.2, page 352): "Let R be a Riemannian manifold homeomorphic with an m-ellipsoid $E_m(a)$ for which $a_1> \cdots >a_{m+1}$. If the constants (a) are sufficiently near unity, there exists a set G of closed extremals which is topologically related on R to the principal ellipses on $E_m(a)$, and which has a kth type number sum at least as great as the number of principal ellipses of $E_m(a)$ of index k." At the end of the chapter come two continuation theorems on geodesics the first of which may roughly be stated as follows: If R_α is a 1-parameter family of Riemannian manifolds and K a critical set of closed geodesics on R_{α_0}, then, if $|\alpha-\alpha_0|$ is sufficiently small and not zero, the manifold R_α contains a finite ensemble K_α of critical sets of closed extremals which "tends to" a subset of K as α tends to α_0 and for which the sum of the kth type numbers is not less than for K.

The account which has been given of the contents of this book is of necessity very sketchy, particularly as far as the last two chapters are concerned; and no mention has been made of work which has appeared since the publication of this volume. This inadequacy is not wholly due to limitations of space, but it is to a large extent the result of the reviewer's limitations; the book makes considerable demands upon the reader. But it is a fascinating and stimulating book. It is to be expected that its reading will lead many mathematicians to engage in the study of the problems which are suggested; the extensive bibliography will be helpful in initiating him in the literature.

The book represents a signal contribution to mathematics. The author is to be heartily congratulated on its completion and the American Mathematical Society deserves our thanks for having made its publication possible.

Very few misprints have come to the reviewer's notice, none of serious character. One desideratum has to be mentioned: the use of the book would be greatly facilitated by an index of the large number of special terms which are used and which are of such character as not to be readily retained by a reader who is not himself working in the field. It is to be hoped and expected that a second edition of the book will soon furnish the opportunity for supplying this need.

ARNOLD DRESDEN

SUPPLEMENTAL REPORT ON ALGEBRAIC GEOMETRY

Selected Topics in Algebraic Geometry, II, Bulletin of the National Research Council, Number 96. Supplemental Report of the Committee on Rational Transformations, Virgil Snyder, Chairman. Published by the National Research Council of the National Academy of Sciences, Washington, D. C., November, 1934. 84 pp.

Bulletin number 96 supplements and brings up to date the original report of the Committee on Rational Transformations entitled, *Selected Topics in Algebraic Geometry*, Bulletin of the National Research Council, Number 63.* That report was so excellent and filled such a pressing need that the Committee was requested by the Council to compile a supplemental report to cover the decade following the period included in the original.

The work of preparing this supplemental report is largely that of the Chairman of the Committee, Virgil Snyder, assisted by Dr. Amos H. Black and Dr. Leaman A. Dye.

The form of this report is similar to that of the original. The discussion is contained in six independent sections, called chapters, of which four are written by Snyder and one each by Black and Dye.

Neither this nor the original report is intended to cover completely the field of algebraic geometry. The title *Selected Topics in Algebraic Geometry* suggests this clearly. Also, as the name of the Committee indicates, the reports are built around the rational transformation. Although the scope of the report is wider than the Committee originally planned, it is still definitely and intentionally limited.

The original report deals with seventeen topics and the supplemental report contains only six. This reduction in the number of topics is due both to omission and concentration.

As stated in the preface, since the recent article, *Algebraische Transformationen und Korrespondenzen*, by Berzolari in the Encyklopädie der mathematischen Wissenschaften has so nearly covered the work of the past decade in the transcendental theory and the mapping of algebraic surfaces, these topics are purposely omitted in this supplemental report. The topics omitted are treated in the original report in Chapters 12, 13, 15, 16, 17.

Evidently there was no attempt to relate definitely the material of the remaining twelve chapters of the original to that of the six chapters of the present report. However, this presents no real difficulty, as the subject matter of each chapter is well described in the contents of that report. The supplemental report also treats some subjects not included in the original.

In the supplemental report, the stated contents of the first and sixth chapters indicate an unsystematic arrangement in the treatment of ruled surfaces that does not actually occur. The title of the first chapter should be simply *Curves*, since, in that chapter, ruled surfaces are treated very briefly and only

* For a review of Bulletin No. 63, see this Bulletin, vol. 38 (1932), pp. 7–11.

incidentally as related to curves. The discussion of ruled surfaces as such occurs in §2 of Chapter 6.

Also, it might have removed a possible impression of lack of continuity in the arrangement of the material if the subdivisions of the reports had been called *Topics*, as they are in the title, instead of *Chapters*, as they are in the subdivision headings. In most books containing consecutively numbered chapters, each chapter leads to the next. A reader of either report might expect this if he had not read the prefaces, particularly that of the original report in which it is stated, "each chapter is to be regarded as a unit" and "any chapter can be read almost independently of the others."

It should be apparent from the great number of papers cited in these reports that the compilers could not have examined critically all the methods used in deriving the results described therein. With each stated result, there is given the author's name and the reference to the memoir in which the result is obtained. Such citations serve two purposes: first, to enable the reader, to whom a certain result (stated with necessary brevity) is not readily intelligible, to go to the original source where it is discussed in detail; and, second, to indicate that the author cited is credited with this result and is the sole and responsible authority for it.

The good features of the original report are continued in this supplement. Each reference is characterized briefly and clearly and the name of the author given with a superscript referring to the citation. These concise descriptions of the contributions of the respective authors are sufficient, in general, to acquaint the reader with the content of the recent literature of the topics treated. Methods are not given in detail, but are often given in outline.

At the end of each chapter is found a list of the references dealt with in that chapter, each preceded by a number which occurs as a superscript after the author's name in the place where the reference is discussed. A total of 791 books and papers by 454 different authors are cited. The omission of one useful feature that occurred in all but four chapters of the original report, cross references to the section containing the discussion of a paper by a given author, is to be regretted.

There is no index, but the titles of section headings are given in the contents, enabling the reader to find readily the discussion of any given subject. The contents are followed by an alphabetical list of periodicals with abbreviations used in the references. Page xii contains a useful list of books on algebraic geometry published since 1924, arranged alphabetically by authors.

This supplemental report adds greatly to the usefulness and scope of the original. A grand total of 3585 books and papers are discussed and cited in the two bulletins, chiefly on rational transformations and associated topics. These two reports constitute a monumental reference work, by far the most important in the field of algebraic geometry.

Those who prepared this supplemental report, as well as the writers of the original, especially the chairman of the Committee on Rational Transformations, Virgil Snyder, merit the thanks and appreciation of every mathematician. Chairman Snyder and his collaborators have given generously of their time in order that these references in algebraic geometry may be rendered more accessible to workers in that field.

A brief description of the contributions of each author of this supplemental report follows.

VIRGIL SNYDER: CHAPTERS 1, 3, 4, 5

Chapter 1. *Curves and ruled surfaces*, pp. 1–13. The generation, construction, associated correspondences, linear series, and residuation theorems of algebraic curves are treated. Plane curves play a leading role, partly because the problem of the existence of plane curves with assigned characteristics has been seriously attacked in the past few years. Ruled surfaces, generated by trisecants or tangents of one curve or by lines associated with two curves, comprise a comparatively small portion of this chapter.

Chapter 3. *Systems of lines in S_n; irregular surfaces*, pp. 25–36. This chapter contains a discussion of generalized complexes, ruled surfaces, and other systems in S_n, criteria for regularity of surfaces and varieties, involutions on surfaces in S_n, and the generation of curves, surfaces, and varieties.

The citation 72.1 is a comprehensive treatment of the highly important problem of the irrationality of the cubic primal in S_4 and related problems, itself containing brief characterizations of 112 references by 45 different authors.

Chapter 4. *Cremona transformations*, pp. 37–62. This chapter deals with cremona transformations of the plane and higher space, associated correspondences, reduction and classification of surfaces, basis elements, series of composition in S_3, invariants of surfaces and varieties, polarities in S_4, and transformations connected with line complexes in S_r. Montesano's multiplier is generalized for S_n and a three-space generalization of the four types of plane involutions is discussed.

Chapter 5. *Multiple correspondences*, pp. 63–72. Involutions of order n and (m, n) correspondences, chiefly in the plane and space, are treated, with some extensions to hyperspace. These correspondences are discussed as to methods of generation, irreducibility, and relations to systems of curves and surfaces.

L. A. DYE: CHAPTER 2

Chapter 2. *The mapping of systems of curves*, pp. 14–24. The mapping of systems of curves on a variety was begun by Klein in 1872. By "mapping" is meant establishing a (1, 1) correspondence between the curves of the system and the elements generating the variety. In the discussion, systems are classified according to the order of their component curves. Systems of lines, conics, cubics, quartics, and special curves, chiefly of S_3, are mapped on varieties of S_n. These systems of curves include both linear and non-linear systems.

A. H. BLACK: CHAPTER 6

Chapter 6. *Surfaces and varieties*, pp. 73–84. The chapter opens with a discussion of cubic and quartic surfaces and ruled surfaces in S_3. The finding of definite criteria (as affirmed by Wiman, references 96, 97) determining the completeness of the classification of sextic ruled surfaces is especially important. It is noteworthy that, if Wiman's criteria are valid, all types of sextic ruled

surfaces (scrolls) were obtained thirty years ago by Snyder, who also stated in his final paper,* "The list of types here found, together with those published in the preceding papers, is believed to be complete."

In the discussion of quintic and sextic (non-ruled) surfaces, the algebraic verification and extension of Montesano's synthetic results for the quintic is of importance.

Under the title *Configurations, surfaces and variations in* S_n, are found six pages of interesting material, difficult to classify. These results largely concern singularities of surfaces and varieties, generation of loci, and generalizations of known properties. The latter part of this section includes some results in differential geometry.

ERRATA

p. v. In title of Chapter I, first word should be "Curves."

p. 33. First part of line 3 from bottom should read, "to n given S_{n-2} in S_n is a ruled V_{n-1}^{n-1}."

p. 48. In bottom line, superscript should be 235 instead of 158.

p. 54. In line 7 from bottom, superscript should be 39 instead of 35.

p. 59. Reference 84 is omitted. This reference is, O. Franceschi, Il Bollettino di Matematico, Firenze, (2), vol. 8 (1929), pp. 137–140. See p. 42, line 11 (from bottom) of report.

p. 60. The author of paper 116 is Hodge (see p. 38, line 7). The name given with 116 belongs with 117.

In addition to the above errata, there are several slight typographical errors especially in superscripts. In general, superscript errors occasion no serious difficulty since author's names accompany respective reference descriptions in the context of each chapter and any name can be readily located in the alphabetical list immediately following that chapter. There is ambiguity, however, when the author has more than one paper in the list. Two cases in which this occurs are included in the errata.

T. R. HOLLCROFT

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* V. Snyder, *On the forms of sextic scrolls having no rectilinear directrix*, American Journal of Mathematics, vol. 27 (1905), p. 173.

SHORTER NOTICES

The Application of Statistical Methods to Industrial Standardization and Quality Control. By E. S. Pearson. London, British Standards Institution, Publication Department, 1935. 161 pp.

The main object of this publication is to set out as clearly as possible why statistical technique is needed and what particular statistical methods are of practical assistance in the solution of certain problems in the industrial standardization or quality control of manufactured articles or of other articles offered on a market. In the fields of standardization and quality control of manufactured products by statistical methods, W. A. Shewhart of the Bell Laboratories was first to show the practical value of these methods. Improvements in the original technique advocated by Shewhart are presented in the book under review. The essential problem is one of testing a batch or consignment including so many items that the practical procedure is to test by sample. In other words, the problem is one in which statistical theory is to be used in estimating the quality of the whole by testing an appropriate but relatively small part of the whole. The concepts of random sampling, representative sampling, statistical uniformity, and level of control are developed and certain principles are formulated in this connection.

Specifications are necessary to characterize the items considered and to indicate the degree of conformity to a standard. An essential part of the specifications is a definition of the method by which conformity is to be secured. Two principal methods are described: (1) that of testing units sampled from consignments, (2) that of requiring records to be kept by the producer which, supplemented by a system of check testing, provides statistical evidence of the level of control maintained during manufacture. The publication deals with the simpler methods of statistical analysis that may be applied during the process of manufacture to assess the level of the control of quality.

It is shown how the results may be usefully presented in the form of simple control charts. Illustrative examples are given throughout the publication to introduce the general problems of inquiry by sample. Then sampling methods in theory and practice are briefly but carefully discussed. Next a procedure is outlined by which the variation of mean and standard deviation in samples are used to estimate the variation of mean and standard deviation of the population. The reliability of a mean estimated from representative samples and from duplicate samples is considered. To obtain, for assigned odds, the upper and lower bounds of the intervals within which the population mean, standard deviation, and coefficient of variability will lie, on the basis of estimates from a sample, it seems desirable in some cases to obtain the fiducial or confidence limits instead of depending on the older "probable error" theory involving the assumption that the number of items in the sample is large. The dependability of the confidence limits depends on the validity of the assumptions:

(1) that the variation is statistically uniform;

(2) that the form of variation among the units is approximately normal;
(3) that the sample is drawn from a supply with at least twenty times as many units as are in the sample.

The basis of control charts is briefly described, with special reference to the methods of estimating the standard deviation of the population. It seems to the reviewer that this publication adapts recent progress in sampling theory admirably to a wide range of applied problems in quality control.

H. L. RIETZ

Einführung in die Liniengeometrie und Kinematik. By Ernst August Weiss. Berlin and Leipzig, Teubner, 1935. vi+122 pp.

This is volume 41 of Teubner's Mathematische Leitfäden and is written by Professor Weiss of the University of Bonn, a pupil of E. Study, the notable geometrician who made the same university a well known center of mathematical activity. The contents of the little text, as Weiss states, are closely connected with the name of Study and it is one of the author's aims to introduce the reader to Study's methods.

After discussion of the linear complex in R_3, line space is mapped upon R_5 as first conceived by Klein. Then follows an investigation of manifolds of complexes and the loci of points and planes connected with them; next an account of Lie's line-sphere transformation and applications in kinematics. The principal theorems of movements and reflexions are derived by the elegant calculus of biquaternions. An important feature of the book is the use of complex spaces and hyperspaces.

The whole treatment is very interesting and affords the reader an insight into a number of novel ideas, which may act as stimulants for workers in this field. The reviewer, however, is under the impression that for the purpose in view and the results obtained, the analytical machinery is in places unnecessarily artificial.

ARNOLD EMCH

Arithmétique et Géométrie sur les Variétés Algébriques. By André Weil, 16 pp.

Quelques Propriétés des Variétés Algébriques se Rattachant aux Théories de l'Algèbre Moderne. By Paul Dubreil. Actualités Scientifiques et Industrielles, Nos. 206, 210. Exposés mathématiques publiés à la mémoire de Jacques Herbrand, Nos. XI, XII. Paris, Hermann, 1935. 33 pp.

These two booklets belong to a series of memoirs published by friends to commemorate Jacques Herbrand, the young French mathematician whose premature death was a severe loss for modern French mathematics. The present pamphlets both deal with the general polynomial ideal theory of E. Noether and van der Waerden, and its applications to algebraic problems. Weil continues investigations by Siegel and himself on the general theory of diophantine equations and shows how certain algebraic decomposition theorems correspond to number theoretical decompositions. Dubreil first studies some properties of the decomposition of homogeneous ideals into primary components and these results are then used to derive general theorems on algebraic varieties.

OYSTEIN ORE

Die Pseudosphäre und die Nichteuklidische Geometrie. By F. Schilling. Parts I and II. 2d edition. Leipzig and Berlin, Teubner, 1935. iv+215 pp.

A review of the first part and first edition of this book appeared in this Bulletin, vol. 38 (1932), p. 335, and what was stated there concerning Schilling's *Pseudosphäre* also holds for the second edition with its addition of a second part.

In the latter the author treats of the geodesic circles of the pseudosphere and their neighborhood (Umwelt). Altogether it contains an admirable account of the geometry on this remarkable surface and its representations of hyperbolic non-euclidean geometry. Particular additions to the usefulness of the book are the beautiful figures of curves and models of the surface. Professor Schilling is a master of graphic representation and has succeeded by more or less elementary means, although in a mathematically rigorous fashion, in making an otherwise abstract subject very attractive.

ARNOLD EMCH

Mengenlehre. By Felix Hausdorff. 3d edition. Berlin and Leipzig, de Gruyter, 1935. 307 pp.

Reviews of the two earlier editions of this work have already appeared in this Bulletin.* The present edition consists of the first nine chapters of the second edition, unchanged except for the correction of misprints and the addition of footnote references to Chapter 10 and the appendices. A new supplementary chapter (Chapter 10, pp. 276–297) has been added, containing two topics on which considerable new material has been published. The first topic is the Baire condition for sets and for functions, the treatment of the text being based on the researches of Banach and Kuratowski. The second half of the chapter is concerned with Borel and Souslin sets, the results being chiefly due to Lusin. Four brief appendices (pp. 298–299) are related to topics discussed in the first nine chapters. The bibliography and source list have also been brought up to date. This revision makes Hausdorff's book even more valuable for those interested in the topics of which it treats.

H. M. GEHMAN

Generalized Hypergeometric Series. By W. N. Bailey. Cambridge Tract, No. 32. Cambridge University Press, 1935. 108 pp.

This tract furnishes an excellent systematic account of the present status of the theory of the generalized hypergeometric series. Since many of the recent contributions to the theory are due to Professor Bailey himself, one can rest assured of having an authoritative treatment of the subject.

The ordinary hypergeometric series is treated in the first chapter, but only to the extent necessary for use in the subsequent discussion. Two of the last three chapters deal with Heine's basic hypergeometric series and Appell's hypergeometric functions of two variables.

C. N. MOORE

* First edition (*Grundzüge der Mengenlehre*) reviewed by H. Blumberg, vol. 27 (1920–21), pp. 116–129; second edition, reviewed by H. M. Gehman, vol. 33 (1927), pp. 778–781.

NOTES

Field Medals of the International Congress of Mathematicians at Oslo were awarded to Professor K. A. M. Ahlfors, professor of mechanical engineering at the School of Technology at Helsingfors, and Dr. Jesse Douglas, professor of mathematics at the Massachusetts Institute of Technology.

Professor R. Nevanlinna, of the University of Helsingfors, was given an honorary degree at the five hundred fiftieth anniversary celebrated at Heidelberg. He was the only mathematican to be so honored. Dean G. D. Birkhoff, of Harvard University, was among those in attendance.

The following mathematicians will receive honorary degrees from Harvard University on September 18: Professors E. J. Cartan, of the University of Paris; L. E. Dickson, of the University of Chicago; G. H. Hardy, of the University of Cambridge, England; and Tullio Levi-Civita, of the University of Rome.

Dr. A. E. Kennelly, professor emeritus of electrical engineering at Harvard University and the Massachusetts Institute of Technology, received the honorary degree of doctor of engineering on the occasion of the centenary of the Darmstadt Institute of Technology.

Professor John von Neumann, of the Institute for Advanced Study, gave four lectures at the University of Paris in May.

Dr. Rufus Oldenburger, of the Armour Institute of Technology, visited some of the European technical schools this summer and gave a paper at the algebra section of the International Congress of Mathematicians at Oslo.

Dr. Harlow Shapley, director of the Harvard College Observatory, has been elected a member of the Norwegian Academy of Sciences.

An honorary degree was conferred on Professor Stepan Timoshenko, of the University of Michigan, by Lehigh University.

Professor Hermann Weyl, of the Institute for Advanced Study, has been elected a member of the Royal Academy of Sciences of Amsterdam and the Royal Society of London.

Professor Edwin Feyer, of the University of Breslau, has been appointed to a professorship at the University of Shanghai.

Dr. H. R. Hulme has been appointed lecturer in the department of applied mathematics at the University of Liverpool in succession to Dr. G. C. McVittie, who has been elected to a readership at King's College, London.

Dr. Karl Kraus, of the Technical School at Brünn, has been appointed to a professorship in mechanics at the Prague Technical School.

Dr. A. G. Walker has been appointed lecturer in pure mathematics at the University of Liverpool in succession to Dr. Haslam-Jones who has been elected a fellow of Queen's College, Oxford.

Mr. L. J. Adams has been appointed head of the department of mathematics at Santa Monica Junior College.

Dr. R. L. Anderson has been promoted to an assistant professorship at Hunter College.

Dr. L. M. Blumenthal, formerly National Research Fellow, has been appointed to an assistant professorship at the University of Missouri.

Dr. W. M. Borgman, of Wayne University, has been promoted to an assistant professorship in mathematics.

Associate Professor C. T. Bumer, of Kenyon College, has been promoted to a professorship in mathematics.

Dr. G. F. Cramer has been promoted to an assistant professorship at Tulane University.

Mr. H. F. Fehr, of the State Teachers College, Montclair, N.J., has been promoted to an assistant professorship.

Dr. K. W. Folley, of Wayne University, has been promoted to an associate professorship.

Dr. J. J. Gergen, of the University of Rochester, has been appointed to an associate professorship at Duke University.

Dr. R. D. James has been promoted to an assistant professorship at the University of California.

Dr. W. I. Miller, of the University of Pittsburgh, has been appointed assistant professor of mathematics at Bucknell Junior College.

Dr. D. C. Morrow, of Wayne University, has been promoted to an associate professorship.

Dr. S. B. Myers has been appointed an assistant at the Institute for Advanced Study.

Dr. Helen G. Russell, of Wellesley College, has been promoted to an assistant professorship in mathematics.

Assistant Professor J. C. Trainor has been promoted to an associate professorship at Washington State Normal School.

At the University of Pennsylvania, Visiting Professor Hans Rademacher has been appointed to an assistant professorship, and Assistant Professors P. A. Caris and J. A. Shohat have been promoted to associate professorships. Professor F. H. Safford retired at the close of the academic year 1935–36.

The following appointments to instructorships have been announced: Brown University: Dr. J. S. Frame; Harvard University: Dr. H. M. MacNeille (Benjamin Peirce Instructor); University of Maine: Dr. S. H. Kimball; Ohio State University: Dr. T. H. Southard; University of Pennsylvania: Dr. J. A. Clarkson; Sophie Newcomb Memorial College: Dr. M. Gweneth Humphreys; Trinity College: Dr. T. L. Downs; Wayne University: Dr. Max Coral; University of Wisconsin (Extension Division): Dr. L. A. Wolf; Yale University: Dr. Nelson Dunford, Mr. J. W. Wrench.

Dr. Wilhelm Kohlrausch, professor of theoretical physics at the University of Strassburg, died recently.

The death of Professor Emma L. Konantz, of Yenching University, Peking, China, is reported.

The death of V. B. Naik, Librarian of the Indian Mathematical Society, and professor of mathematics at the Fergusson College, Poona, is reported.

The death of Dr. W. A. Parr, President of the British Astronomical Association, is reported.

The death of Professor Ludwig Stickelberger, of the University of Freiburg, is reported.

President Ellen F. Pendleton, of Wellesley College, died on July 26, 1936.

Mr. G. A. Plimpton, of New York City, died on July 1, 1936, at the age of eighty years.

Dr. F. C. Touton, of the University of Southern California, died on June 1, 1936.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THE SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

318. Dr. J. S. Frame: *On the degrees of the irreducible components of simply transitive permutation groups.*

In a simply transitive permutation group G^* of order g and degree n, the subgroup leaving one symbol invariant will permute the others in λ sets of transitivity involving $k_1, k_2, \cdots, k_\lambda$ symbols respectively. If this group G^* be represented in the usual way by a group G of matrices of degree n, in which the matrix $(\delta_{\alpha'\beta})$, $\alpha, \beta = 1, \cdots, n$, corresponds to the permutation $\binom{\alpha}{\alpha'}$, the group will be a reducible linear group with $r = \lambda + 1$ irreducible components, having the respective degrees $n_0 = 1, n_1, n_2, \cdots, n_\lambda$. Let $N = n_1 n_2 \cdots n_\lambda$, and $K = k_1 k_2 \cdots k_\lambda$. The object of this paper is to show that $n^{\lambda-1}K/N$ is an integer, which is a perfect square when the numbers k_i are distinct. We prove this theorem for $\lambda \leqq 3$, and also for an infinite family of permutation groups (including all values of λ), which consist of the permutations induced by the symmetric group of degree ν on the $n = \binom{\nu}{\lambda}$ sets of ν symbols taken λ at a time. A somewhat complicated identity in binomial coefficients is proved as a lemma. (Received July 25, 1936.)

319. Professor W. L. Ayres: *Periodic transformations of sets.*

Periodic, pointwise periodic, and almost periodic transformations of sets are defined and shown to have unique inverses. A pointwise periodic continuous transformation of a graph is shown to be periodic. If the graph contains at least one branch point, pointwise periodic may be replaced by almost periodic. If any of these transformations on a tree leaves the end points invariant it leaves all points invariant. Either a pointwise or almost periodic continuous transformation of a tree is periodic if only a finite set of end points are non-invariant. (Received July 27, 1936.)

320. Dr. Reinhold Baer: *Abelian groups without elements of finite order.*

In this paper the properties of direct sums of rational groups and of separable groups are investigated (rational group = subgroup of the additive group of rational numbers; separable group = group whose finite subsets are contained in direct summands which are direct sums of rational groups). There exist

separable groups which are not direct sums of rational groups (e.g., the additive group of all the sequences of integers) but every countable separable group is a direct sum of rational groups. Further criteria for the existence of a decomposition into rational groups and for separability are given. Though every Abelian group without elements of finite order is contained in a direct sum of rational groups, it is possible to characterize a class of subgroups which are isomorphic with a direct summand and are themselves direct sums of rational groups. Finally the structures of direct sums of rational groups, the types of elements, and the characteristic subgroups of separable groups are enumerated. Only invariant properties are used in the mentioned criteria and enumerations. The applied method is essentially a calculus with multiplicities where the multiplicity of the element x in the group J is the l.c.m. of the positive integers n such that $x \equiv 0 \bmod nJ$. (Received July 22, 1936.)

321. Professor D. H. Ballou: *Functions representable by two Laplace integrals.*

Functions $f(z)$ are considered which are single-valued, meromorphic, and simply-periodic, and hence developable into series both of partial fractions and of exponentials. By considering the nature and position of the poles in a period strip classes of these functions are found which can be expressed as two different Laplace integrals. From the equality of these integrals there then follow as corollaries the equalities of the series of their integrands. The meromorphic trigonometric and hyperbolic functions are specific examples of this theory. (Received July 6, 1936.)

322. Dr. E. F. Beckenbach: *On a theorem of Fejér and Riesz.*

Let the analytic function $w=f(z)$ map $|z| \leqq 1$ conformally on a region R. The map on R of a diameter of $|z| \leqq 1$ will be called a conformal diameter of R. According to a theorem of Fejér and Riesz, a conformal diameter of R can be at most half as long as the perimeter of R. The result now is extended, by use of subharmonic functions, to conformal diameters of regions on surfaces of negative Gauss curvature. The average of the lengths of the conformal diameters of such regions also is discussed. (Received July 27, 1936.)

323. Mr. Garrett Birkhoff: *Linear representability of nilpotent Lie algebras.*

It is shown that every Lie algebra can be realized by infinite matrices with coefficients in the same abstract field, and that every nilpotent Lie algebra can be realized by finite triangular matrices. It follows that every nilpotent Lie algebra with real or complex coefficients is the infinitesimal algebra of a simply connected group of matrices. (Received July 23, 1936.)

324. Professor Richard Brauer: *Hypercomplex numbers and factor sets.*

A new method is developed for deriving the properties of factor sets. The basis for the considerations is formed by Wedderburn's theorems. Thus a new approach to some parts of the theory of algebras is obtained. (Received July 24, 1936.)

325. Professor Richard Brauer: *On algebras which are connected with the simple continuous groups.*

Let G be a group of linear transformations in an n-dimensional vector space R_n. The tensors of given rank f undergo a homomorphic group of linear transformations when the transformations of G are performed in R_n. The matrices of these tensor transformations generate an algebra A_f the study of which is of great importance for the theory of representations of G. It will be shown that the construction of a basis of A_f and the proof of the fundamental theorem of invariant theory imply each other. If G is the full linear group, A_f can be easily investigated in a direct way and thus we obtain a proof of the fundamental theorem of invariant theory. In case G is the orthogonal group or the complex group we go the opposite way. We thus obtain the algebra A_f. In case G is the complex group this algebra has been constructed by Weyl (Mathematische Zeitschrift, vol. 35) by using a quite different method. (Received July 24, 1936.)

326. Professor Richard Brauer: *On algebraic equations of the fifth and sixth degrees.*

The solution of equations of the fifth and sixth degrees is equivalent to the solution of a form problem for the icosahedron and the Valentiner group, as was shown by Klein, Gordan, Coble, and Fricke. The same question is here dealt with again by using a general theory for form problems already given in a previous paper (Mathematische Annalen, vol. 110). (Received July 24, 1936.)

327. Professor R. S. Burington: *R-matrices and equivalent networks.*

In recent papers the author has developed a theory relating to m-affine congruent matrices and their application to the study of networks. If B is a matrix of a linear m-terminal pair network and T is any real m-affine non-singular matrix, then $A = T'BT$ corresponds to a network (electrically) equivalent to B. The range V of T for which networks corresponding to A are realizable without ideal transformers is desired. In this paper a theory of R-matrices is developed, a real matrix $G \equiv (g_{rs})$ being termed an R-matrix if and only if $g_{ss} \geqq 0$ and $\sum_{r=1}^{n} |g_{rs}| \leqq 2g_{ss}$, $s = 1, \cdots, n$. Necessary and sufficient conditions are given that $C = P'DP$ be an R-matrix, where P and D are real and P is non-singular. With each D there exist n sets of functions $X^{(i)}$, each set forming a group of order 2^{n-1}. The $n \cdot 2^{n-1}$ quadric surfaces $X^{(i)} = 0$ bound a region in the space of P for which C, positive definite, is an R-matrix. Application is made to network theory, necessary and sufficient conditions being found, together with V, for realizability without ideal transformers of networks corresponding to A. (Received July 23, 1936.)

328. Professor L. P. Eisenhart and Professor M. S. Knebelman: *Invariant theory of homogeneous contact transformations.*

We define *a configuration* as an entity whose components undergo a certain linear homogeneous transformation when the element (x, p) is subjected to a homogeneous contact transformation. We introduce into our space a *contact*

frame $\Gamma_{\alpha\beta}(x, p)$ by means of which a configuration is given a unique set of tensor components. We then introduce a linear connection—subject to a set of symmetry conditions—by means of which covariant differentiation is defined. It is by this means that we can attack the problem of equivalence of configurations under contact transformations. We apply these ideas to the concepts of parallelism, geodesics, and curvature. (Received July 29, 1936.)

329. Mr. R. M. Foster: *A simplified set of postulates for a group.*

A simplified set of postulates for a finite group has been given by Garver (this Bulletin, vol. 42 (1936), pp. 125–129). It is shown in the present note that Garver's simplification is not essentially limited to the special case of finite groups, that the same simplification can be effected in the set of postulates for a general group. (Received August 1, 1936.)

330. Professor L. M. Graves: *On the uniform boundedness of sets of functional transformations between topological spaces.*

In this paper we consider certain generalizations of well known theorems (see Banach, *Opérations Linéaires*, p. 19, Theorem 11; Hildebrandt, this Bulletin, vol. 29 (1923), p. 309) to the case when the transformations involved have for domain a topological space and for range a linear topological space (see von Neumann, Transactions of this Society, vol. 37 (1935), p. 4). (Received July 27, 1936.)

331. Mr. O. H. Hamilton: *A property of hereditarily continuous curves.*

It is shown that, in any locally compact space satisfying Axioms 0, 1, 2, 3, and 4 of R. L. Moore's *Foundations of Point Set Theory*, if H and K are mutually exclusive closed point sets, there does not exist a compact hereditarily continuous curve M (that is, a continuous curve, every subcontinuum of which is a continuous curve) which contains an infinite number of mutually exclusive continua, each containing a point of H and a point of K. (Received July 16, 1936.)

332. Professor G. A. Hedlund: *Fuchsian groups and transitive horocycles.*

The region under consideration is the interior, Ψ, of the unit circle, U, provided with the Poincaré metric. The horocycles are curves of constant geodesic curvature unity and are euclidean circles internally tangent to U. Denoting by F a Fuchsian group with principal circle, U, and considering points congruent under F identical, we obtain a two-dimensional manifold, M, of constant negative curvature, its combinatorial topological properties being determined by F. In this paper it is shown that if F is of the first kind (nowhere properly discontinuous on U) there exist transitive horocycles; that if F has a fundamental region lying, together with its boundary, interior to U, all horocycles are transitive; that if F is of the first kind and has a finite set of generators the horocycles are transitive except for a denumerable set. These results

make it possible to show that the flow defined by the geodesics on M is asymptotically transitive and permit applications to automorphic functions. This paper will appear in the Duke Mathematical Journal. (Received July 20, 1936.)

333. Professor G. A. Hedlund: *Two-dimensional manifolds and transitivity.*

Morse has shown (Journal de Mathématiques, vol. 14 (1935), pp. 49–71) that if the geodesics on a closed orientable surface, S, of genus greater than one are uniformly unstable, there exists a non-denumerable infinity of transitive geodesic rays passing through any point of the surface. By making use of results known in the case of constant negative curvature the fundamental result of Morse can be extended in two senses: (1) the hypothesis of uniform instability can be slightly weakened and (2) the surfaces under consideration can include cases of infinite genus and surfaces with singularities. This paper will appear in the Annals of Mathematics. (Received July 20, 1936.)

334. Dr. Charles Hopkins: *Concerning uniqueness-bases of finite groups with applications to regular p-groups of class* 2.

An ordered set of elements $P_1, P_2, \cdots, P_r$ is said to constitute a uniqueness-basis for a finite group G if every element of G can be expressed uniquely in the form $P_1^{x_1}P_2^{x_2}\cdots P_r^{x_r}$, $0 \leqq x_i < g_i$, where g_i is the order of P_i. Several sufficient conditions are established for the existence of a uniqueness-basis in an arbitrary finite group; and it is proved that every regular p-group G' has an ω-normal uniqueness-basis: that is, a uniqueness-basis $P_1, P_2, \cdots, P_r$, the order of P_i being p^{δ_i}, such that (1) $\delta_1 \geqq \delta_2 \geqq \cdots \geqq \delta_r$; (2) in the commutator $(P_j, P_i) = P_1^{\alpha_{ji1}}\cdots P_r^{\alpha_{jir}}$, $i<j$, each exponent α_{jik} is divisible by p for $k \leqq j$. Under the assumption that G' is of class 2 a method is developed for obtaining a true representation of G' by $(r+1)$-rowed square matrices. (Received July 22, 1936.)

335. Professor Dunham Jackson: *Orthogonal polynomials on a plane curve.*

In a recent paper (abstract 42-1-18) the writer has given an exposition of some of the fundamental properties of polynomials in two real variables orthogonal over a region of the (x, y)-plane. The present paper is concerned with relations of correspondingly fundamental character for polynomials orthogonal on a curve in the plane. The distinction between algebraic and non-algebraic curves gives rise to considerations of some interest which are not encountered in the case of a two-dimensional region. (Received July 24, 1936.)

336. Professor B. W. Jones: *On the reduction of positive quaternary quadratic forms.* Preliminary report.

If the coefficients of a positive quaternary quadratic form b are a_{ij}, $i, j=1, 2, 3, 4$, and $a_{ii} = -(a_{i1}+a_{i2}+a_{i3}+a_{i4})$ it has been shown that f is equivalent to a form of one of three types: all $a_{ij} \leqq 0$, $a_{ij} > 0$ for just one pair ij, $a_{ij} > 0$ for just two pairs which do not have a subscript in common (where in each of the

types $i, j=1, 2, 3, 4, 5, i \neq j$). Conditions are found which, for forms of the first type, define a *unique* reduced form. These conditions are of two kinds: first, inequalities like $a_{11} \geqq a_{22}$ which are not symmetric; second, conditions like: if $a_{ip}=a_{ij}=0$, then $|a_{in}|\alpha|a_{im}|$ and $|a_{pn}+a_{jn}|\beta|a_{pm}+a_{jm}|$, where one of α and β is $=$, both are $<$, or both are $>$. The second kind of condition is symmetric to the extent that it is independent of any permutations of the subscripts. (Received July 25, 1936.)

337. Dr. P. W. Ketchum: *On certain infinite systems of linear equations and an expansion problem of Pincherle.*

Expansions in series of the form $\sum c_n F_n(z)$, where $F_n(z)$ is a given analytic function of the form $g_n(z)+\sum_{i=1}^{\infty}\alpha_{n,i}g_{n+i}(z)$ have been studied extensively for for the case $g_n(z)=z^n$ by Pincherle and numerous later writers. We show that these expansion theorems possess analogues in the theory of the solution of certain systems of linear equations in an infinity of unknowns. Conversely, these results are used to obtain generalizations of the above expansions to cases where $g_n(z)$ no longer has the special form z^n, but may, for instance, have n zeros at a certain set of points $\beta_{n,i}$, $i=1, 2, \cdots, n$. (Received July 27, 1936.)

338. Professor Otto Laporte: *Stereographic parameters and pseudo-minimal hypersurfaces.* II.

In a previous paper it was shown that to every pseudo-minimal hypersurface there belongs a solution of a certain simple linear elliptic partial differential equation. Such solutions were called pseudo-minimal potentials and an infinity of special potentials having singularities at either the point zero or the point infinity were obtained. In this paper displaced solutions, having singularities at arbitrarily given points, are constructed through an application of the group of the differential equation. These solutions are symmetric in the coordinates of the singularity and of the point under consideration. The second part of the paper is concerned with the possibility of constructing one-sided pseudo-minimal hypersurfaces in generalization of the classical researches of Lie, Henneberg, and others. It is seen that for a pseudo-minimal hypersurface to be one-sided the potential has to fulfill a certain functional equation. Linear combinations of the particular solutions given in the previous paper are easily constructed which satisfy the functional equation. The above developments hold in spaces of any number of dimensions. (Received July 29, 1936.)

339. Dr. Walter Leighton: *On the convergence of continued fractions.*

Let $1+K_1^{\infty}[b_n/1]$ be a continued fraction in which the b_n are complex numbers $\neq 0$. Further, let $m_0, m_1, m_2, \cdots$ be any sequence of positive numbers such that $m_0=2$, $m_{n+1}-m_n \geqq 2$ $(n=0, 1, 2, \cdots)$. The numbers $|b_{m_0}|$, $|b_{m_1}|$, $|b_{m_2}|, \cdots$ can be chosen in such a fashion that excluding at most one value in the complex plane from each of the other b_n's, the continued fraction will converge. (Received July 18, 1936.)

340. Dr. Walter Leighton: *A test-ratio test for continued fractions.*

This paper presents what is believed to be the first test-ratio test for continued fractions. Let $1+K_1^{\infty}[b_i/1]$ be a continued fraction in which the b_i are complex numbers $\neq 0$. If the ratio $|b_{i+1}/b_i| \leqq k<1$ for i sufficiently large, the continued fraction converges at least conditionally. It diverges, if ultimately $|b_{i+1}/b_i| \geqq 1/k>1$. If $\lim_{i=\infty}|b_{i+1}/b_i| = 1$, the question of convergence is indeterminate. (Received July 24, 1936.)

341. Dr. Saunders MacLane: *When can a graph be mapped on a torus?*

This paper gives a strictly combinatorial condition that a given combinatorial graph be mappable on a torus by generalizing a method previously used for planar graphs (abstract 42-5-195-t). The method consists essentially in combinatorially characterizing the boundaries of complementary regions of a graph on a torus. First, a non-separable graph G of nullity n can be mapped on a torus if it contains a set S of $n-1$ oriented circuits such that: (1) each directed arc of G is contained in just one circuit of S; (2) any two arcs a and b both ending on a vertex q are connected by a chain of arcs $a=a_1, a_2, \cdots, a_m=b$, such that each a_i is "adjacent" to the succeeding arc a_{i+1}. Here two arcs a_i and a_{i+1} are "adjacent" if they belong to one of the circuits of S. In the second place, a non-separable graph can also be mapped on the torus if it is a planar graph or can be made planar by "splitting" one of its vertices in two. Conversely, it is then shown that any non-separable graph on a torus must satisfy one of the two combinatorial conditions stated above. This necessary and sufficient condition is extended to separable graphs. (Received July 25, 1936.)

342. Dr. H. M. MacNeille: *A normal form for the extension from a distributive lattice to a Boolean ring.*

The elements of the extension, L, to a Boolean ring of a distributive lattice, K, can be represented by finite descending sequences of elements of K, whether the extension be made by means of the symmetric difference or by Boolean addition. If K has completely distributive products, a property which is extensionally obtainable from any distributive lattice, then K is shown to be a homomorph of L under a correspondence which preserves the ordering relation and Boolean sums of L but not, in general, products or complements. By means of this homomorphism, a representative is selected from each class of equal elements in L without employing the axiom of choice. The set of these representative elements is a normal form for the extension of K to a Boolean ring. (Refer to the Proceedings of the National Academy of Sciences, vol. 22 (1936), pp. 45–50.) (Received July 28, 1936.)

343. Professor Morris Marden: *On Kakeya's problem.* Preliminary report.

This is the problem of determining the least $R>0$ such that, for every polynomial of degree n having p $(2 \leqq p \leqq n)$ zeros in or on the unit circle, the deriv-

ative has at least $p-1$ zeros in or on the circle $|z|=R$. The number R has been found for $p=n$ by Gauss and Lucas, for $p=2$ by Alexander, Kakeya, and Szegö, and for $p=n-1$ by Biernacki. In the present note it is proved that $R \leqq \csc [\pi/2(n-p+1)]$ by using a lemma that if $\alpha_1, \alpha_2, \cdots, \alpha_p$ are any $p (2 \leqq p \leqq n)$ distinct zeros of a polynomial $f(z)$ of degree n and if $\zeta_1, \zeta_2, \cdots, \zeta_q$ ($\zeta_j \neq \alpha_k$, all j and k) are any $q=n-p+1$ distinct zeros of $f'(z)$, then $\sum A^{-1}_{k_1 k_2 \cdots k_q}(\zeta_1-\alpha_{k_1})^{-1}(\zeta_2-\alpha_{k_2})^{-1}\cdots(\zeta_q-\alpha_{k_q})^{-1}=0$, the sum being taken for $k_1, k_2, \cdots, k_q = 1, 2, \cdots, p$ and $A_{k_1 k_2 \cdots k_q}$ being equal to the number of different permutations of the q numbers $k_1, k_2, \cdots, k_q$ taken q at a time. The author hopes both to sharpen the limits given and to extend the results to higher derivatives. (Received July 26, 1936.)

344. Professor W. H. McEwen: *An extension of Bernstein's theorem associated with general boundary value problems.*

This paper deals with sums $S_n(x)=\sum_{k=1}^{*}(a_{1k}u_{1k}(x)+a_{2k}u_{2k}(x))$, in which the u_{ik}'s are characteristic solutions of $L(u)+\lambda u=0$, $U_j(u)=0$ $(j=1, 2, \cdots, m)$, a given mth order linear differential system with regular boundary conditions on $a \leqq x \leqq b$, and the a_{ik}'s are arbitrary constants. If the maximum of $|S_n(x)|$ on (a, b) be denoted by L, it is shown that (1) $|S_n'(x)| \leqq qn^2 L$ on $a \leqq x \leqq b$, and (2) $|S_n'(x)| \leqq QnL$ on $a+\epsilon \leqq x \leqq b-\epsilon$, q and Q being constants independent of n, the latter depending on ϵ. Applications of these results are made to certain least rth power problems of best approximation. (Received July 9, 1936.)

345. Professor E. J. McShane: *Semi-continuity of integrals in the calculus of variations.*

A theorem on the semi-continuity of single integrals of the calculus of variations is established under hypotheses weak enough so that the ordinary problem, the parametric problem, the Lagrange problem in ordinary and in parametric form, and the parametric problem associated with an ordinary problem all are contained as special cases. In addition, this theorem enables us to establish the existence of minimizing curves for several problems (for example, the Zermelo navigation problem) not covered by theorems in the literature. (Received August 3, 1936.)

346. Professor A. D. Michal: *The tensor calculus in a general Riemannian differential geometry.*

In this paper the author continues his investigations on general Riemannian geometry with Banach coordinates. The laws of transformation of the abstract Christoffel symbol and its adjoint are derived and the resulting absolute differential calculus is then studied with the aid of the author's more general absolute calculus in non-Riemannian geometry. (Received July 27, 1936.)

347. Professor A. D. Michal, Dr. I. E. Highberg, and Dr. A. E. Taylor: *Abstract euclidean spaces.* V: *Rotations in spaces with indefinite geometrical metric.*

The work of Michal and Elconin on abstract transformation groups with abstract parameters is used to characterize rotation groups in several func-

tional spaces with an indefinite geometrical metric by means of completely integrable differential equations in Fréchet differentials. Tonelli's recent work on non-linear functional equations is employed in the proof of the existence of non-linear rotations in a complete functional space with an indefinite geometrical metric. (Received July 27, 1936.)

348. Professor A. D. Michal and Mr. D. H. Hyers: *Theory and applications of abstract normal coordinates in a general differential geometry.*

One of the authors has initiated the study of general differential geometry with a linear connection in which the customary arithmetic coordinates of n-dimensional geometry are replaced by elements of a Banach space (A. D. Michal, Proceedings of the National Academy of Sciences, vol. 21 (1935), pp. 526–529, and several forthcoming papers in Acta Mathematica, Annali di Matematica, this Bulletin, and Compositio Mathematico). In this paper we define abstract normal coordinates and prove their existence with the aid of a theorem on second-order differential equations in Banach spaces. This theory does not make use of any abstract power series expansions so that many of the recent results of Veblen, Whitehead, and T. Y. Thomas on normal coordinates in n-dimensional geometry may be obtained as special cases of our theorems. Theories of abstract "normal tensors" and of extensions of abstract tensor forms are developed. A theory of adjoints and their differentials, although trivial in the n-dimensional case, plays an important role here and in all infinitely-dimensional instances. In case the Banach space is a space of continuous functions, this paper gives new results in a Michal functional geometry. (Received July 27, 1936.)

349. Professor C. N. Moore: *On convergence factors in restrictedly convergent double series.*

The origin and subsequent development of the notions of restricted convergence and restricted summability of multiple series have been outlined in the abstract of a paper presented at the last annual meeting in St. Louis (abstract 42-1-34). In the present paper necessary and sufficient conditions are found for convergence factors in restrictedly convergent double series. For convergence factors of type I (Transactions of this Society, vol. 29 (1927), p. 227) no modifications are needed in the necessary and sufficient conditions for the case of series convergent in the general sense. For convergence factors of type II (l.c.) two supplementary conditions are required. In the first of these conditions it is stipulated that the convergence factors $f_{ij}(\alpha)$, defined over a set $E(\alpha)$ with a limit point α_0, not of the set, be such that the upper limit, as α approaches α_0, of $\sum_{j=0}^{\infty}\sum_{i=0}^{[\theta j]}|\Delta_{11}f_{ij}(\alpha)|$ tend to zero as θ approaches zero. The other condition is the analogue of this, with the roles of i and j interchanged. The results indicated include as special cases two theorems due to Lösch (Mathematische Annalen, vol. 110 (1934), pp. 33–53; Theorems I and II). (Received July 14, 1936.)

350. Professor F. D. Murnaghan: *A symmetric reduction of the planar three body problem.*

If a, b, c denote the sides of the triangle formed by the three bodies, of masses m_1, m_2, m_3, and θ_1, θ_2, θ_3 the inclinations of the sides to a fixed line in the plane of the three bodies, the dynamical system has four degrees of freedom when the center of mass is taken as the origin of an inertial reference frame and a, b, c, and $\phi=\frac{1}{2}(\theta_1+\theta_2+\theta_3)$ may be taken as the coordinates of the system; ϕ is ignorable and if k is the corresponding momentum (angular) constant, a simple calculation yields the following expression for the modified (Routh) Hamiltonian function H^*: $H^*=\sum(1/(2m_1))[\omega_2^2+\omega_3^2+2(\omega_2\omega_3+k^2/(bc))\cos A+2k(\omega_2/b-\omega_3/c)\sin A+k^2(1/b^2+1/c^2)]-U$, where ω_1, ω_2, ω_3 are the momenta corresponding to the coordinates a, b, c and $U=-r[(m_2m_3)/a+(m_3m_1)/b+(m_1m_2)/c]$ is the force function. (Received July 24, 1936.)

351. Professor F. D. Murnaghan: *On universally valid stress-strain relations for an elastic solid possessing an energy of deformation.*

The general relations connecting the stress and strain tensors are derived without making the hypothesis that the strain is infinitesimal. The classical relation $P^{rs}=\partial\phi/\partial\epsilon_{rs}$, where P is the stress and ϵ the strain tensor, is shown to hold only for infinitesimal strains; the generally valid formula which reduces to the above classical relation when the strain is infinitesimal is given. For an isotropic medium it is $P^{rs}=\partial\phi/\partial\epsilon_{rs}-2\epsilon_a^r(\partial\phi/\partial\epsilon_{rs})$. A numerical application to the theory of metals under high hydrostatic pressures and the theory of Young's modulus for large extensions is given. (Received July 24, 1936.)

352. Mr. F. J. Murray: *Complementary manifolds in L_p and l_p, $1<p\neq 2$.*

If Λ is a Banach space, $\mathfrak{M}$ a closed linear manifold in Λ, a closed linear manifold $\mathfrak{N}$ such that every $f\epsilon\Lambda$ is uniquely expressible in the form $g+h$, $g\epsilon\mathfrak{M}$, $h\epsilon\mathfrak{N}$, is called a complementary manifold to $\mathfrak{M}$. In his treatise on *Linear Operations*, Banach presents the following problems. (a) To every $\mathfrak{M}$ in L_p does there exist a complementary manifold? (b) To every $\mathfrak{M}$ in l_p does there exist a complementary manifold? In this paper it is shown that the answer to both questions is "No." (Received July 24, 1936.)

353. Professor John von Neumann and Dr. I. J. Schoenberg: *Fourier integrals and metric geometry.*

Let $\phi(t)$ $(-\infty<t<\infty)$ be a continuous function with functional values in a metric space γ with the distance function $d(\phi,\psi)$. If $d(\phi(t),\phi(t'))=\sigma(t-t')$ is a function of the difference $t-t'$ only, we call the curve Γ: $\phi(t)$ a screw curve (s.c.) of γ and $\sigma(t)$ a screw function (s.f.) of γ, since Γ admits the one-parameter group $t\to t+\tau$ of rigid displacements within itself. W. A. Wilson (American Journal of Mathematics, vol. 57 (1935), pp. 63–64) has shown that $\sigma(t)=|t|^{1/2}$ is a s.f. of Hilbert space $\mathfrak{Z}$. We prove: (i) The most general s.f. of $\mathfrak{Z}$ is given

by the equation $\sigma^2(t)=\int_0^\infty(\sin tu/u)^2d\gamma(u)$, where $\gamma(u)(u\geqq 0)$ is non-decreasing and such that $\int_1^\infty u^{-2}d\gamma(u)$ exists. (ii) The most general s.f. of the unit-sphere of $\mathfrak{Z}$ is given by the equations $\cos\sigma(t)=g(t)$, $0\leqq\sigma(t)\leqq\pi$, where $g(t)=\int_0^\infty\cos tud\alpha(u)$, $\alpha(u)$, non-decreasing, is a real positive definitive function with $g(0)=1$. The s.f. of euclidean spaces and finite-dimensional spheres are obtained, if $\gamma(u)$ and $\alpha(u)$ have a finite number of points of increase only. (iii) Obviously (ii) must be a special case of (i). It is characterized by the finiteness of $\int_0^\infty u^{-2}d\gamma(u)$, or, equivalently, by the boundedness of $\sigma(\tau)$. (ii) may readily be derived from known results; (i) and (iii) necessitate an investigation of the continuous one-parameter linear groups of rigid displacement of $\mathfrak{Z}$ within itself. (Received July 20, 1936.)

354. Professor Gordon Pall: *Applications of the rational automorphs of* $x^2+y^2+z^2$. Preliminary report.

The theory of certain relations among the solutions of $n=x^2+y^2+z^2$ is studied. The known formula for $r_3(nm^2)/r_3(n)$ is derived by a simple method. New properties of integral quaternions are obtained. (Received July 25, 1936.)

355. Mr. E. W. Paxson: *The integral in abstract linear topological spaces*. Preliminary report.

This paper continues the discussion of the analysis of linear non-metric topological spaces instituted with studies of the differential by A. D. Michal and the author. (See Comptes Rendus, vol. 202 (1936), pp. 1741–1743, and abstract 42-5-137 and a paper offered to the Fundamenta Mathematicae.) The "Riemann" integral is defined and developed for functions on the real numbers to the space L. The usual classical theorems of manipulation are verified. Some important topological facts for the linear space L are established with the aid of which the existence of a unique primitive solution (the indefinite integral) for the equation $F'(\mu)=f(\mu)$ is demonstrated. (Received July 27, 1936.)

356. Dr. Hillel Poritsky: *A problem of replacement.*

Suppose that a new installation requiring for its operation a constant number N_0 of units of a particular article—say thyratron tubes—is put into operation at the time $t=0$. The replacement problem leads to the integral equation $r(t)=N_0p(t)+\int_0^t r(s)p(t-s)ds$, where $r(t)$ is the probable rate of replacement, and $p(t)$ is the life expectancy of a unit, that is, the probability of a unit failing (a tube burning out) after it has been in operation a time t. More convenient than the classical Volterra solution is the operational method of solution which consists in replacing $r(t)$, $p(t)$ in the above equation by Laplace or Carson integrals. After solving the resulting equation for $\rho(p)$, we determine r by the Bromwich integral. Of practical interest is the case $p(t)=1$ for $0<t<h$ and $p(t)=0$ elsewhere. In this case the above integral equation may be replaced by a difference-differential equation. The operational method readily yields an expansion for $r(t)$ in terms of the characteristic functions of this equation. Similar solutions are obtainable for cases where $p(t)$ is a step function or is represented by polynomials in successive intervals. (Received July 25, 1936.)

357. Dr. Hillel Poritsky: *On finger print patterns and their singular points.*

This note contains a study of finger (also of palm) print patterns, particularly in regard to the singular points of these curve families and their indices. The index for each finger segment is zero. At the terminal segments there is one singular point with a positive index of either 1/2 or 1. In the former case there is also one singular point of index $-1/2$; in the latter case there are two such points. A speculation on the mode of origin of the patterns connecting them with lines of maximum curvature is discussed. (Received July 25, 1936.)

358. Dr. Hillel Poritsky: *Thermal stresses in cylindrical tubes.*

The problem considered is that of stresses due to a two-dimensional steady state temperature distribution. The elastic and thermal coefficients, as well as the coefficient of expansion k are assumed to be constant. After a uniform expansion and bending have been allowed for, a solution of the thermoelastic equations is given by $w=0$, $u+iv=\int\phi(x+iy)d(x+iy)$, where (u, v, w) are the displacement components in the x, y, z directions, and ϕ is the analytic function whose real part is kT, T being the temperature. For solid cylinders the above solution is satisfactory and results only in axial stresses. For pipes, however, owing to the multiple connectivity, the above displacement is multiple-valued and increases by a rigid rotation for a path enclosing the hole. The correcting state of stress, known as a "dislocation," is determined, and expressions are derived for the Airy's functions corresponding to these dislocations, in terms of analytic functions. (Received July 25, 1936.)

359. Professor Hans Rademacher: *Dedekind's ζ-function and Hecke's $\zeta(s, \lambda)$-functions in totally real algebraic fields.*

The sum $\Phi(a, s)=\sum_{\mu}N(a+i\mu)^{-s}$ for real $a_1, a_2, \cdots, a_n>0$, $R(s)>1$, where μ runs through all integers of a given totally real algebraic field, is treated in two different ways, one of which includes a Fourier development with respect to the point lattice of all integers of the field. Comparison of both results yields an identity which contains the functional equations of the Dedekind ζ-function and of all Hecke $\zeta(s, \lambda)$-functions simultaneously. (Received July 27, 1936.)

360. Professor Hans Rademacher: *On Waring's problem in algebraic fields.*

Hardy-Littlewood's method has already been applied to certain problems of additive number theory in algebraic fields. But Waring's problem in algebraic fields still offered a particular difficulty, arising from the "minor" Farey sections, since no analogue of Weyl's method of approximation was known. In the present paper this analogue is developed for totally real, in particular, real quadratic, fields. The generalized finite geometric series to which by Weyl's procedure a certain sum of roots of unity is reduced necessitates a Fourier expansion which can be avoided in the rational case. (Received July 27, 1936.)

361. Dr. W. T. Reid: *Sufficient conditions by expansion methods for the problem of Bolza in parametric form.*

In a recent paper (abstract 41-11-379) the author has obtained, using the expansion method introduced by E. E. Levi, sufficient conditions for a strong relative minimum in the non-parametric problem of Lagrange in the calculus of variations. The expansion method is used in the present paper to prove a sufficiency theorem for a strong relative minimum in the problem of Bolza in parametric form. The hypotheses of the theorem do not involve any assumptions of normality, and the results of this paper are essentially the same as those recently obtained by Hestenes [American Journal of Mathematics, vol. 58 (1936), pp. 391–406] using the classical field construction methods of the calculus of variations. (Received July 27, 1936.)

362. Mr. H. E. Robbins: *On a class of recurrent sequences.*

Recurrent sequences have been defined by Birkhoff (*Dynamical Systems*, p. 246) and Morse (Transactions of this Society, vol. 22 (1921), p. 95). The present paper contains a definition of a class of recurrent, non-periodic sequences which prove the existence of a r.n.p. sequence whose ergodic function is bounded above by a linear function, and of r.n.p. sequences whose ergodic functions increase arbitrarily rapidly. (Received July 27, 1936.)

363. Dr. R. M. Robinson: *The theory of classes: a modification of von Neumann's system.*

The theory of classes presented in this paper is a simplification of that presented by J. von Neumann in his paper *Die Axiomatisierung der Mengenlehre* (Mathematische Zeitschrift, vol. 27 (1928), pp. 669–752). As in his paper, *function* rather than *class* is taken as the fundamental idea; the terms *function* and *argument*, and the operator $[f, x]$ (value of the function f for the argument x), are primitive concepts of the system. In von Neumann's paper, the idea *ordered pair* was an additional primitive concept; here it is defined in terms of the others. Also, the present paper gives a much simpler proof of the well-ordering theorem based on von Neumann's *equivalence* axiom, which stated roughly is: *A function is an argument if and only if it is defined for fewer arguments than there are arguments in all.* The central idea of the proof is a simple characterization of ordinal numbers, which makes it unnecessary to develop a theory of ordering. (Received July 9, 1936.)

364. Professor Robin Robinson: *A condition in invariant form for a net without detours.*

The term *net without detours* is used here as a translation of the German *Kurvennetz ohne Umwege*. A necessary and sufficient condition for such a system of curves on a surface has been given by Rothe (Sitzungsberichte der Berliner Mathematischen Gesellschaft, 7, (1908), pp. 14–15). The condition given in this note is equivalent to that of Rothe, but it is written in invariant form, and is derived in an entirely different manner. (Received July 6, 1936.)

365. Professor I. M. Sheffer: *A simplified solution of equation* $\Delta y = F(x)$.

We have elsewhere considered, from the point of view of a local solution, the equation (1) $\Delta y = F(x)$ and certain related equations. It has now been found possible to simplify the form of the solution of (1) by altering the method of solution. This is the purpose of the present note. It is believed that the new method may be effective for other equations as well, although this has not as yet been carried out in detail. (Received July 25, 1936.)

366. Professor L. L. Silverman: *On products of Nörlund transformations.*

Inasmuch as the ordinary product of two Nörlund transformations is in general not of the Nörlund type of summability it has been found desirable to introduce a new definition of product. These two definitions of product are not in general equivalent. The object of the present paper is to study the relations of these two definitions of product of Nörlund transformations. (Received July 6, 1936.)

367. Professor A. E. Staniland: *Analytic affine transformations in euclidean space of 2n dimensions.*

A euclidean space, S_{2n}, is employed as a real representation of a complex linear n-space, Σ_n, the real point $(x_1, x_2, \cdots, x_{2n})$ corresponding to the complex point $(z_1, z_2, \cdots, z_n)$, with $z_k \equiv x_{2k-1} + ix_{2k}$. Subjecting the real coordinates to the general affine transformation and insisting that the generalized Cauchy-Riemann equations continue to hold for all complex functions initially analytic we obtain necessary and sufficient conditions that the transformation be analytic. The analytic affine transformations of S_{2n} correspond exactly to the complex affine transformations of Σ_n, $A: z_k = \sum A_{kj} z_j' + B_k$ and $\bar{A}: \bar{z}_k = \sum A_{kj} z_j' + B_k$, $|A_{kj}| \neq 0$. The unitary transformation U of type A reduces to $U = T^{-1} U_0 T$ where T, unitary of type A, is used to effect a change to new coordinate analytic planes, and $U_0 = \pi R_k$, $R_k: z_k = e^{i\phi_k} z_k'$, $z_h = z_h'$, $h \neq k$ $\bar{U}$ is obtained from U by replacing z_k by $\bar{z}_k$. The A_{kj} are exhibited as rational functions of the complex parameters, together with their conjugates, which determine the planes, and as transcendental functions of the ϕ_k. Moreover, ϕ = amplitude $|A_{kj}| = \sum \phi_k$. (Received July 24, 1936.)

368. Mr. Alvin Sugar: *Ideal Waring theorem for the polynomial* $m(x^3 - x)/6 - m(x^2 - x)/2 + x$.

As a consequence of a theorem obtained in a previous paper (to appear soon in the American Journal of Mathematics) by the author, we know that every positive integer is a sum of $m+3$ values of $G(x) = m(x^3 - x)/6 - m(x^2 - x)/2 + x$ for $m \geq 16$. This result, however, is not ideal. In this paper we prove the ideal theorem for $G(x)$: *Every positive integer is a sum of* $[(m+1)/2] + 3$ *values of* $G(x)$ *for* $m \geq 42$. (Received July 27, 1936.)

369. Dr. A. E. Taylor: *A generalization of the Cauchy-Riemann equations.*

In two previous reports on analytic functions in general analysis (abstracts 41-11-424 and 42-3-147) it is shown that a large portion of the Cauchy-Weierstrass theory of analytic functions may be generalized for functions on one complex vector space to another, the second space being complete. The foregoing results may then be applied to pairs of "conjugate" functions of two variables ranging over a *real* vector space, by forming a complex space of couples, in exactly the same way that the complex numbers are formed by pairs of reals. As a consequence we obtain necessary and sufficient conditions for pairs of conjugate functions in the form of generalized Cauchy-Riemann equations in Fréchet differentials. (Received July 20, 1936.)

370. Dr. C. C. Torrance: *On product-continuous sets.*

The object of this paper is to define, and to develop some of the properties of, product-continuous sets in topological spaces. (Received July 28, 1936.)

371. Professor W. J. Trjitzinsky: *Non-linear difference equations.*

In the present work an investigation is given of the analytic character of solutions (in the neighborhood of the singular point $x=\infty$) of the non-linear nth order difference equation $y(x+n)=x^{p/q}a(x, y(x), y(x+1), \cdots, y(x+n-1))$ (integers $p, q; q\geqq 1$), the function $a(x, z_0, z_1, \cdots, z_{n-1})$ being analytic in $z(=x^{1/q})$, $z_0, z_1, \cdots, z_{n-1}$ at $(z=\infty, z_0=z_1=\cdots=z_{n-1}=0)$ while $a(x, 0, 0, \cdots, 0)\equiv 0$. At the basis of this work lies a new method due to W. J. Trjitzinsky for treatment of non-linear problems (in the fields of difference and differential equations) and introduced by this writer for the first time in a memoir on differential equations which will appear in the Mémorial des Sciences Mathématiques. (Received July 23, 1936.)

372. Professor A. W. Tucker: *Mappings and products of cell spaces.*

Following a previous paper (Annals of Mathematics, vol. 37 (1936), pp. 92–100) which dealt with cell spaces and their closed and open mappings, this paper shows how these mappings may be characterized in terms of products. A closed mapping $S\rightarrow S'$ of one cell space on another is represented on the product $S^*\times S'$ (S^* = dual of S) by a sub-set whose left and right closures are identical. If S and S' are complexes, the algebraic counterpart of a mapping is represented by a chain on $S^*\times S'$; this chain is a cycle when the mapping is closed. If $S\rightarrow S'$ is a single-valued mapping and S' has closure uniformity, the common left and right closure mentioned above is an "expanded" image of S^*. S itself has as subdivided image a sort of intersection complex derived from $S\times S'^*$. Open mappings lead to dual considerations. (Received July 28, 1936.)

373. Professor H. S. Wall: *On hypergroups.*

The concept of hypergroup given in an earlier communication is enlarged. The product of two elements is a complex of n elements of the system. Here n

is a fixed integer $\geqq 1$. If $n=1$ the hypergroup is an ordinary group. The associative law holds; there is at least one element e such that ea and ae contain a (at least once) for every a; corresponding to an arbitrary element a there is at least one element a^{-1} such that aa^{-1} and $a^{-1}a$ contain e. Certain hypergroups are related to the Mischgruppe of Loewy and Baer. They include a large class of multiplicative systems derived from an ordinary group. About thirty-five theorems are given, many of which are analogs of theorems for ordinary groups. (Received July 21, 1936.)

374. Professor H. S. Wall: *On the nth derivative of $f(x)$.*

This note gives explicitly the coefficient of $ff_1^a f_2^b \cdots f_n^c, (a+2b+\cdots+nc=n)$, in the polynomial expression for $f^{(n)}(x)$ in terms of f and $f_1, f_2, \cdots, f_n$, the first n derivatives of $\log f(x)$. The coefficient is $n!/(a!b!\cdots c!(1!)^a(2!)^b\cdots(n!)^c)$. (Received July 21, 1936.)

375. Professor Morgan Ward: *Arithmetical functions on rings.*

Let $\mathfrak{o}$ be a ring, Σ a structure (lattice), and ϕ a function on $\mathfrak{o}$ to Σ. ϕ is arithmetical if ϕ_a divides ϕ_b in Σ whenever a divides b in $\mathfrak{o}$. This paper presents a general investigation of the properties of various types of arithmetical functions which clarifies and extends earlier work of the author and others on linear recurring series and divisibility sequences. As a sample result, let S be any element of Σ, and assume that the aggregate of elements s of $\mathfrak{o}$ such that S divides ϕ_s form an ideal $\mathfrak{s}=\rho_s$ of $\mathfrak{o}$, "the rank of apparition of S in ϕ". Then ρ is an arithmetical function on Σ to $\mathfrak{o}$. If furthermore A, B are any two divisors of ϕ with ranks of apparition $\mathfrak{a}$, $\mathfrak{b}$, and if the cross-cut M of A, B is also a divisor of ϕ, then the cross-cut of $\mathfrak{a}$, $\mathfrak{b}$ is ρ_M. (Received July 25, 1936.)

376. Dr. J. F. Wardwell: *Concerning separating transformations which are entirely complete.*

A continuous transformation $T(A)=B$ which is not (1-1) will be called separating provided that every set $T^{-1}(b)$, $b \epsilon B$, separates A. Such a transformation T will be called completely separating if every $T^{-1}(b)$, $b \epsilon B$, separates every $T^{-1}(p)$, for $p \epsilon B-b$, in A. Finally, any completely separating transformation T will be called entirely complete provided that, for any possible separation $A-T^{-1}(b)=A_1+A_2$, $T^{-1}(p)\cdot A_1 \neq 0$ and $T^{-1}(p)\cdot A_2 \neq 0$, for every $p \epsilon B-b$. This paper deals mainly with the latter type of transformation. If A is a compact continuum and $T(A)=B$ is continuous, some necessary and sufficient conditions in order that T be entirely complete are obtained. Furthermore if T is entirely complete, it is shown that B will contain no cut points. Hence, if A is locally connected, B will be cyclicly connected. It is also shown that, in this case, A will contain at least one true cyclic element and, under certain conditions, will contain some true cyclic element E_a such that $T(E_a)=B$. Among various other results in this paper are those obtained by the application of such transformations to some particular curves and surfaces. (Received July 27, 1936.)

377. Mr. A. H. Wheeler: *A general formula for polyhedra with applications to higher dodecahedra and icosahedra.*

In a previous paper (see Proceedings of the International Mathematical Congress, Toronto, 1924, vol. I, pp. 701–708) the author defined and classified high icosahedra by means of lines bounding the faces of the solids. The purpose of the present paper is to provide a general formula which will completely define all polyhedra independently of relations of symmetry, as in the previous paper. Applications are readily made to higher forms of solids, including those having two or more kinds of faces as in the case of the Archimedean solids. (Received July 22, 1936.)

378. Professor Hassler Whitney: *On regular closed curves in the plane.*

Let C be a closed curve with continuously turning tangent, with any singularities. Let γ be the total angle through which the tangent turns while traversing the curve. It is shown (theorem and a different proof suggested by W. C. Graustein) that two curves with the same γ may be deformed into each other. If one follows the curve, starting from an "outside point," and counts the numbers N^+ and N^- of times that one crosses a part of the curve already traced in the "positive" and "negative" sense, then $\gamma = 2\pi[\mu + (N^+ - N^-)]$, where $\mu = \pm 1$ according to the orientation of the curve. (Received July 20, 1936.)

379. Mr. Reinhard Korgen: *An extension of the concept called language.*

This paper distinguishes between "articulate" and "inarticulate" *languages*, and describes as an example of an inarticulate language a "language of materials" which does not require man as an agent for its use. It is used by man *in combination* with articulate languages to form a "language of thinking," and is thus also used by man to communicate ideas. Implications bearing on the foundations of mathematics will be considered. (Received July 27, 1936.)

380. Professor Alonzo Church and Dr. S. C. Kleene: *Constructions of formal definitions of functions of ordinal numbers.*

In an article entitled *Formal definitions in the theory of ordinal numbers* (forthcoming in the Fundamenta Mathematicae) the authors have set up definitions extending the theory of definition to transfinite ordinals and have constructed λ-definitions of some ordinals and functions of ordinals. In the present paper it is shown how the construction can be carried out under more general circumstances. In particular it is proved that Veblen's generalizations of the Cantor ϵ-numbers are λ-defineable (a result previously stated without proof). Further an explicit proof is given that the equality of ordinals is not λ-defineable; also that no function is λ-defineable which associates with every λ-defineable ordinal a unique enumeration of the class of lesser ordinals. A definition of *recursive ordinal*, suggested by Gödel, is introduced, but, probably because of the difficulty of finding a definition of *recursive function of ordinals*, the notion of recursiveness fails to play the role that it does in connection with λ-definition in the theory of positive integers. (Received July 22, 1936.)

381. Mr. E. C. Berkeley: *Two practical problems requiring techniques from symbolic logic.*

This paper outlines two problems arising in group insurance and in beneficiary settlements where methods from symbolic logic would prove of considerable use. (Received July 22, 1936.)

382. Professor Paul Weiss: *The self-contradictory.*

The supposition that the self-contradictory is a proposition, obtainable by substitution and by negating a necessary truth is beset by six difficulties. 1. Necessary truths cannot, by definition, be denied. 2. There are many necessary truths and only one self-contradictory, though each proposition has its own denial. 3. Since $p \cdot q$ might be true and $p \cdot -p$ cannot be true, to substitute $-p$ for q is to obtain a necessary falsehood from a possible truth by a legitimate act of substitution. 4. The disjunction in $p \vee -p$ is different from the disjunction in $p \vee q$, and the conjunction in $p \cdot \vee -p$ must be different from the conjunction $p \cdot q$. Yet there is only one kind of "and" in extensional logic. 5. The self-contradictory negates and yet entails the necessary. But a conclusion must be consistent with its premiss. 6. The antecedent contains the meaning of its consequent. But if a self-contradictory proposition entails a necessary proposition, a part will be incompatible with its whole. A solution is offered in the shape of the contention that negated elements are values of the terms of a necessary truth, and that a so-called self-contradictory has an indeterminate subject and is not a proposition at all. This entails the denial of the generality of the law of transposition, but not of De Morgan's theorem or of the *reductio ad absurdum*. (Received July 22, 1936.)

383. Dr. A. F. Emch: *The calculus of logical implication.*

It is the purpose of this paper to show that "the calculus of logical implication" proposed in the first two issues of The Journal of Symbolic Logic is such that (a) its properties coincide with the properties of deducibility, (b) the paradoxical theorems of strict as well as of material implication are explained, (c) logical and mathematical propositions can be both consistent and independent, and (d) the entire calculi of strict and material implications can be derived from it. The paper is developed under the following rubrics: *Introduction*; §1. *The primitive ideas, definitions, postulates and rules of operation*; §2. *C properties* (theorems which do not include either explicitly or by definition the primitive idea of "possibility"); §3. *The calculus of material implication*; §4. *S properties and the calculus of strict implication* (theorems which do not include, either explicitly or by definition, the primitive idea of "logical consistency" but do include, either explicitly or by definition, the primitive idea of "possibility"); §5. *L properties, consistency and the modal functions* (theorems which include, either explicitly or by definition, both the primitive ideas "logical consistency" and "possibility"); §6. *Existence principles*; and §7. *Conclusion* (in which four optional postulates are suggested for further consideration). (Received July 22, 1936.)

384. Dr. L. O. Kattsoff: *Group theory for truth values.*

Aristotelian logic is now seen to be a logic of two truth values as is also the Russell-Whitehead logic. Lukasiewicz in conjunction with Tarski published in 1930 a set of postulates for a logic of four truth values. Becker has also developed postulates for logic for six and twelve truth values. The indications today are towards an ever-increasing number of truth values. This paper makes a first attempt to develop not the properties of a logic having a given number of truth values, but the properties of the truth values themselves. A truth value is treated as the result of an operation upon a group of symbols. Hence, it is possible to speak either of the operator or of the result of the operation. The concepts of group theory are applied to these operations and certain theorems are proved which indicate two important results. (1) Every logic based on a set of truth values not forming a group is bound to break down. (2) The two-value logic has been the most successful because its truth values do form a group and there are indications that they are a subgroup of a larger group. (Received July 22, 1936.)

385. Professor Paul Henle: *The algebras of concentric sectors.*

Boolean algebra is intimately connected with the number two. It contains 2 unique elements; it has a law of double negation; and the number of its elements, in finite cases, is a power of 2. This paper presents a generalization of these characteristics, yielding a calculus containing n unique elements, a law of n-tuple negation, and containing, in finite cases, a number of elements which is a power of n. When $n=2$, Boolean algebra results; but for each $n\neq 2$, there is a new algebra. All these algebras may be developed in terms of two binary operators, "$+$" and "$\times$". Any law of Boolean algebra composed only of the symbols, "$+$", "$\times$", "$=$", and real variables holds for all of them. They are capable of a spatial interpretation, their elements being represented as sectors or combinations of sectors of concentric circles; the number of concentric circles being determined by n. In every algebra, $a+b$ may be represented by the area included either in a or b and $a\times b$ may be represented by their common area. Each algebra contains a unary function, definable in terms of $+$ and $\times$, corresponding to the negation of Boolean algebra, and it is chiefly in respect to laws involving this function that the algebras differ. (Received July 22, 1936.)

386. Dr. F. B. Fitch: *A formal theory of types.*

Instead of introducing types into a system of formal logic by way of informal conventions or special rules of procedure, a method of introduction is proposed which consists of the mere addition of a finite number of formal axioms. The formal system to which these axioms are added employs material implication, propositional negation, universal, quantification, and operators analogous to the combinatory operators I, B, and C; but it differs from combinatory logic in not employing an analog to the operator W. The effect of the additional axioms is to introduce an infinite sequence of analogs to the W-operator, W^1, W^2, W^3, $\cdots$, such that a function not involving W^m for any $m \geqq n$ would be "of nth type," and such that if θ is of nth type, then

$W^n(\phi, \theta)$ and $\phi(\theta, \theta)$ would be materially equivalent propositions in the logic. The Russell paradox is still avoided; because although a ψ can be found such that, for all θ of nth type, $\sim\theta(\theta)$ would be materially equivalent to $\psi(\theta)$, yet ψ would have to be of higher than nth type. The other paradoxes of Mengenlehre may be analogously treated. Much of number theory seems to be deducible from the resulting formal system. (Received July 22, 1936.)

387. Professor Rudolf Carnap: *Truth in mathematics and logic.*

First problem. What is the nature of the truth of mathematical and logical theorems? We have to distinguish between factual truth and formal truth. A sentence of empirical science is true or false according to whether the (possible) fact asserted by it subsists or not. A sentence of the language of mathematics and logic does not assert a fact. Such a sentence is true or false according as the rules of the language-system state it to be true (analytic) or false (contradictory). *Second problem.* How to construct a system of mathematics, that is, how to define the concept "true mathematical sentence"? (a) The customary *method of demonstration* consists in laying down some primitive sentences (postulates) and some rules of inference (with a finite number of premises). Example: The system of Principia Mathematica. But, according to Gödel's results, such a system is always incomplete. (b) Therefore we are compelled to apply *rules of a new kind* referring to infinite classes of premises. By rules of this kind the concept of the truth of universal sentences concerning integers can easily be defined; but for sentences concerning real numbers, that is, classes of integers, certain difficulties arise. (Received July 22, 1936.)

338. Professor Richard Courant: *Conformal mapping and Plateau's problem.*

The connection between Plateau's problem and the problem of conformal mapping has been emphasized already by Radó and Douglas, the latter of whom has stressed the point that the solution of Plateau's problem should be considered as a natural way to solve problems of conformal mapping.

The present paper intends to show that, by using freely the classical theory of conformal mapping and a certain generalisation of it, one obtains a very simple solution not only of the problem of Plateau, but also of the widest generalisation of it by Douglas and also of the more general problems in which parts of the contours are free to move on prescribed manifolds. The details of this theory (see two Notes in the Proceedings of the National Academy of Sciences for June, 1936) will be published in the Annals of Mathematics together with another solution of the problems of Plateau, independent of conformal mapping. (Received August 24, 1936.)

389. Mr. Philip Hall: *Solvable groups.*

A group of order g is solvable if and only if it has at least one subgroup of order m, for every divisor m of g which is prime to g/m. Let G be solvable of order $p_1^{a_1} \cdots p_r^{a_r}$, ($p_1, \cdots, p_r$ being distinct primes). Then G possesses a "Sylow system" of subgroups $S_1, \cdots, S_r$ with the properties (*i*) S_i is of order

$p_i^{a_i}$, and (*ii*) $S_iS_j=S_jS_i(i, j=1, \cdots, r)$. Any two Sylow systems are conjugate, and the Sylow systems of any subgroup H of G can all be obtained as meets of H with suitably chosen Sylow systems of G. Let $G=G_0>G_1> \cdots >G_k=1$ be any chief series of G. Call G_i/G_{i+1} *central* or *eccentric* according as it does or does not belong to the central of G/G_{i+1}. Then the number of distinct Sylow systems in G is the product of the orders of the eccentric factors G_i/G_{i+1}. More precisely: Let N be the normalizer of any Sylow system $S_1, \cdots, S_r$ of G (that is, the meet of the normalizers of the S_i). Then, if G_i/G_{i+1} is central, the join of N with G_{i+1} contains G_i; while if G_i/G_{i+1} is eccentric, the meet of N with G_i lies in G_{i+1}. If the S_i are all abelian, the meet of N and the derived group G' of G is 1, and their join is G. (Received August 11, 1936.)

390. Dr. A. N. Lowan: *On some two-dimensional problems in heat conduction.*

This paper deals with the problem of heat conduction in an infinite parallelopiped bounded by the planes $y=0$ and $y=b$ which are assumed to be impervious to heat and by the planes $x=0$ and $x=a$ for which the expressions $\alpha\partial T/\partial x+\beta T$ are prescribed functions of y and t, the initial temperature distribution being $f(x, y)$, the solid containing a continuous distribution of heat generating sources described by the function $\Psi(x, y, t)$. The method employed is the two-dimensional extension of that presented in a number of earlier papers. (A. N. Lowen, *Note on the cooling of a radioactive sphere*, American Journal of Mathematics, vol. 56 (1934), No. 2; *Heat conduction in a semi-infinite solid*, American Journal of Mathematics, vol. 56 (1934).) With the aid of the Laplace transform the problem is reduced to the integration of certain ordinary differential equations and the final solution is obtained by the inversion of the Laplace transform, frequent use being made of Borel's theorem and of other theorems in the operational calculus. From the solutions obtained, the corresponding solutions are derived for the cases $a=\infty$ and $b=\infty$, the passage to the limit being accomplished with the aid of the known solutions of some suitably chosen auxiliary problems in heat conduction. (Received August 11, 1936.)

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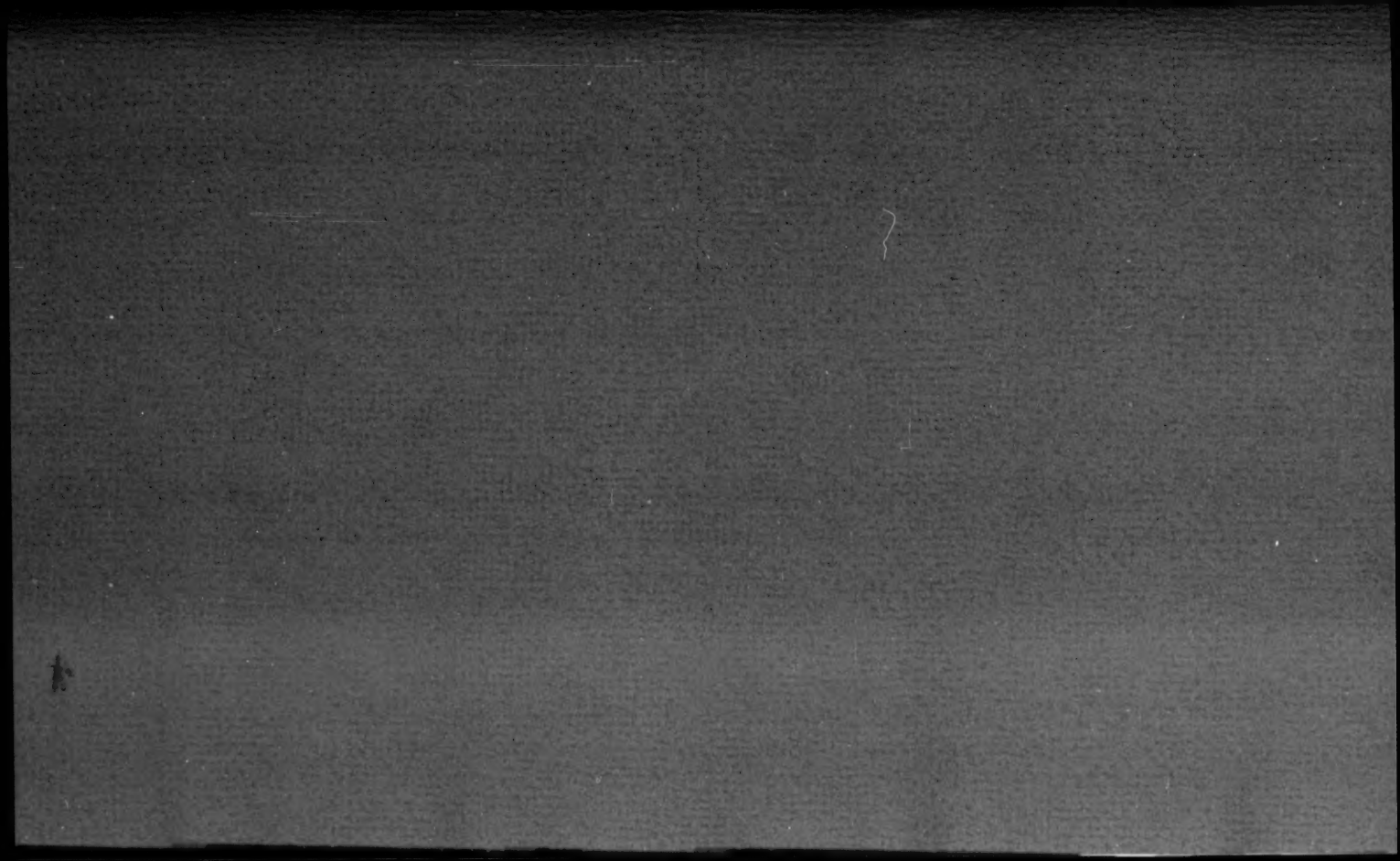
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